Tutorial

21 December 2019

Schwartz space, Tempered Distributions and Fourier Transform

1. The **Schwartz space** $\mathcal{S}$ is the set of $C^\infty(\mathbf{R}^d)$ functions f satisfying $\sup_{x\in\mathbb{R}^d} |(1+|x|^2)^k D^\alpha f(x)| < \infty$ for all multi-indices α and non-negative integers k, endowed with the topology induced from the countably many seminorms given by $|f|_k \coloneqq \sup_{x\in\mathbb{R}^d, |\alpha|\leq k} (1+|x|^2)^k D^\alpha f(x)|$. The Schwartz space $\mathcal{S}$ is a Frechét space, i.e., it is a topological vector space whose topology is induced by a complete metric.

2. Recall that we use the notation $\mathcal{D}(\mathbb{R}^d)$ for smooth compactly supported functions on $\mathbb{R}^d$. Also recall that a linear functional $T : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$ is said to be a **distribution** if for every compact set $K \subset \mathbb{R}^d$ there is a constant $C = C(K) > 0$ and an integer $N = N(K)$ such that $|T(\phi)| \leq C||\phi||_N$ for all $\phi \in \mathcal{D}(\mathbb{R}^d)$ whose support lies in K and $||\phi||_N$ is the sum of the norms of ϕ and all its derivatives upto order N.

3. The dual space of $\mathcal{S}$ is the space of all **tempered distributions** denoted $\mathcal{S}^{'}$. Since $\mathcal{D} \subset \mathcal{S}$, it can be shown that a tempered distribution is also a distribution through its restriction to $\mathcal{D}$.

4. We define the **Fourier transform** for $\phi \in \mathcal{S}(\mathbb{R}^d)$ as

$$\hat{\phi}(\xi) = \mathcal{F}(\phi)(\xi) = \int_{\mathbb{R}^d} \phi(x) e^{-ix\cdot\xi} dx.$$

   Indeed this integral makes sense for $\phi \in L^1(\mathbb{R}^d)$ and consequently for $\phi \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$.

5. Properties of Fourier transform

   (a) Fourier transform maps $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$.

   (b)
$$\mathcal{F}(exp\left(-\frac{1}{2}|x|^2\right)) = (2\pi)^{d/2} exp\left(-\frac{1}{2}|\xi|^2\right).$$

   (c) Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$ isomorphically. The inverse transform is defined by
$$\mathcal{F}^{-1}(\psi)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(\xi) e^{ix\cdot\xi} d\xi$$
   so that
$$\phi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) e^{ix\cdot\xi} d\xi.$$

   (d)
$$\frac{1}{(i)^{|\alpha|}} D_\xi^\alpha(\mathcal{F}(\phi)) = \mathcal{F}((-x)^\alpha \phi)$$

   (e)
$$\mathcal{F}\left(\frac{1}{(i)^{|\alpha|}} D_x^\alpha \phi\right) = \xi^\alpha \mathcal{F}(\phi) \text{ for } \psi, \phi \in \mathcal{S}(\mathbb{R}^d),$$

   (f)
$$\int_{\mathbb{R}^d} \hat{\phi}\psi dx = \int_{\mathbb{R}^d} \phi\hat{\psi} d\xi.$$

(g)

$$\int_{\mathbb{R}^d} \phi\overline{\psi}dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\phi}\overline{\hat{\psi}}d\xi.$$

so that

$$||\psi||^2_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}||\hat{\psi}||^2_{L^2(\mathbb{R}^d)}.$$

6. The last equality allows us to extend the Fourier transform upto $L^2(\mathbb{R}^d)$ as an isometry. In that case, the Fourier transform can no longer be thought of as an integral but as a limits of integrals.

7. The definition of Fourier transform may be extended to tempered distributions in the following manner. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, then $\langle \hat{f}, \phi\rangle := \langle f, \hat{\phi}\rangle$. Most properties of Fourier transforms in the case of Schwartz class functions extend to tempered distributions. In particular, for $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, observe that

$$\langle D^\alpha \widehat{f}, \phi\rangle = (-i)^{|\alpha|}\langle \widehat{x^\alpha f}, \phi\rangle$$

and

$$\langle \widehat{D^\alpha f}, \phi\rangle = (i)^{|\alpha|}\langle \xi^\alpha \widehat{f}, \phi\rangle.$$

Moreover, these equations hold true not only in the sense of distributions but in the strict sense for functions in Sobolev spaces where derivative is defined not only as a distribution. For $u \in H^k(\mathbb{R}^d)$, it holds true that $\widehat{D^\alpha u} = (i)^{|\alpha|}\xi^\alpha \widehat{u}$ in L^2 sense.

Trace Theory

1. In the lectures, the notion of trace was defined for the upper half plane. Using partition of unity, it can be extended to any bounded domain with a sufficiently smooth boundary. In particular, the following theorem holds.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class C^{m+1} with boundary Γ. Then there exists a trace map $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1})$ from $H^m(\Omega)$ into $(L^2(\Omega))^d$ such that*

(a) If $v \in C^\infty(\overline{\Omega})$, then $\gamma_0(v) = v|_\Gamma$,$\gamma_1(v) = \frac{\partial v}{\partial \nu}|_\Gamma$, ..., $\gamma_{m-1}(v) = \frac{\partial^{m-1} v}{\partial \nu^{m-1}}|_\Gamma$, where ν is the unit exterior normal to the boundary Γ.

(b) The range of the map γ is the space $\displaystyle\prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma)$.

(c) The kernel of γ is $H^m_0(\Omega)$.

2. Let Ω be a bounded open set with sufficiently smooth boundary and let $u \in H^1(\Omega)$. Then there exists a sequence (u_n) from $C^\infty_c(\mathbb{R}^d)$ such that $u_n|_\Omega \to u$ in $H^1(\Omega)$. Using this density result and trace theorem, we can extend the Green's formula to functions in $H^1(\Omega)$, i.e., for $u, v \in H^1(\Omega)$ we have

$$\int_\Omega u\frac{\partial v}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i}v + \int_\Gamma (\gamma_0 u)(\gamma_0 v)\nu_i. \tag{1}$$

Similarly, for $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have

$$\int_\Omega \nabla u \cdot \nabla v = -\int_\Omega (\Delta u)v + \int_\Gamma (\gamma_0 v)(\gamma_1 u). \tag{2}$$

Moreover, for $u, v \in H^2(\Omega)$, we have

$$\int_\Omega (\Delta v)u - \int_\Omega (\Delta u)v = \int_\Gamma (\gamma_0 u)(\gamma_1 v) - \int_\Gamma (\gamma_0 v)(\gamma_1 u). \tag{3}$$

Regularity Theory

1. The equation $u'' + u = f$ in (a,b) with $u(a) = u(b) = 0$ has a solution $u \in H_0^1(a,b)$ for $f \in L^2(a,b)$. It can be seen from the weak formulation of the equation that $u \in H^2(a,b)$. By Morrey's inequality, $u \in C^{1,\frac{1}{2}}(a,b)$. If $f \in C[a,b]$, then again from the equation, we obtain $u \in C^2(a,b)$. This is false in higher dimensions. Let $R < 1$ and $B_R(0) = B_R$ the ball in $\mathbb{R}^d$ with center at the origin. Let $x = (x_1, ..., x_d)$ and define

$$f(x) = \frac{x_2^2 - x_1^2}{2|x|^2}\left[\frac{d+2}{(-\log|x|^{1/2})} + \frac{1}{2(-\log|x|)^{3/2}}\right]$$

$$u(x) = (x_1^2 - x_2^2)(-\log|x|)^{1/2}$$

$$\phi(x) = \sqrt{-\log R}(x_1^2 - x_2^2)$$

You can verify that $f \in C(\overline{B}_R)$, $u \in C(\overline{B}_R) \cap C^\infty(\overline{B}_R \setminus \{0\})$. Also,

$$\begin{cases} \Delta u = f & \text{in } B_R \\ u = \phi & \text{in } \partial B_R \end{cases}$$

Nonetheless, $\lim_{|x|\to 0} D_{11}u(x) = 0$, which implies that $u \notin C^2(B_r)$.

2. Case I: $\mathbb{R}^d$ Let u solve the equation $-\Delta u + u = f$ in $\mathbb{R}^d$ where $f \in L^2(\mathbb{R}^d)$. A solution to the weak formulation exists in $H^1(\mathbb{R}^d)$ by Lax-Milgram lemma. We can show that $u \in H^2(\mathbb{R}^d)$ by the following steps:

Step 1. Let $w \in L^2(\mathbb{R}^d)$. For $h \in \mathbb{R}^d$, define $D_h w(x) = \frac{w(x+h)-w(x)}{|h|}$. Prove that $\frac{\partial w}{\partial x_i}$ exists in $L^2(\mathbb{R}^d)$ if and only if $||D_{te_i}w||_{L^2(\mathbb{R}^d)}$ is bounded for all $t > 0$. In fact, $||D_{te_i}w||_{L^2(\mathbb{R}^d)} \leq ||\frac{\partial w}{\partial x_i}||_{L^2}$

Step 2. Taking $D_{-h}(D_h u)$ as a test function in the weak formulation, we obtain the bound $||D_h u||^2_{H^1} \leq ||f||_{L^2}||D_{-h}(D_h u)||_{L^2}$.

Step 3. Using Step 1 on $w = D_h u$, we get $||D_{-h}(D_h u)||_{L^2} \leq ||\nabla(D_h u)||_{L^2}$.

Step 4. Combining the inequalities in Step 2 and 3, we get $||D_h(\nabla u)||_{L^2} \leq ||f||_{L^2}$. Finally using Step 1, we get $\frac{\partial^2 u}{\partial x_i \partial x_k}$ exists in $L^2(\mathbb{R}^d)$ for all $1 \leq i \leq d$ and $1 \leq k \leq d$.

Step 5. Using induction and by "differentiating the equation", we can prove that if $f \in H^m(\mathbb{R}^d)$ then $u \in H^{m+2}(\mathbb{R}^d)$.

3. Case II: $\mathbb{R}^d_+$ Let u solve the equation

$$\begin{cases} -\Delta u + u = f & \text{in } \mathbb{R}^d_+ \\ u = 0 & \text{in } \partial\mathbb{R}^d_+ \end{cases}$$

where $f \in L^2(\mathbb{R}^d_+)$. A solution to the weak formulation exists in $H_0^1(\mathbb{R}^d_+)$ by Lax-Milgram lemma. We can show that $u \in H^2(\mathbb{R}^d_+)$ by the following steps:

Step 1. Let $w \in L^2(\mathbb{R}^d_+)$. For $h \in \{(h_1, h_2, \cdot, h_{d-1}, 0) : h_i \in \mathbb{R}\}$, define $D_h w(x) = \frac{w(x+h)-w(x)}{|h|}$. For $1 \leq j \leq d-1$, prove that $\frac{\partial w}{\partial x_i}$ exists in $L^2(\mathbb{R}^d_+)$ if and only if $||D_{te_i}w||_{L^2(\mathbb{R}^d_+)}$ is bounded for all $t > 0$. In fact, $||D_{te_i}w||_{L^2(\mathbb{R}^d_+)} \leq ||\frac{\partial w}{\partial x_i}||_{L^2(\mathbb{R}^d_+)}$

Step 2. For $1 \leq j \leq d-1$, taking $D_{-te_i}(D_{te_i}u)$ as a test function in the weak formulation, we obtain the bound $||D_{te_i}u||^2_{H^1} \leq ||f||_{L^2}||D_{-te_i}(D_{te_i}u)||_{L^2}$.

Step 3. Using Step 1 on $w = D_{te_i}u$, we get $||D_{-te_i}(D_{te_i}u)||_{L^2} \leq ||\nabla(D_{te_i}u)||_{L^2}$ for $1 \leq j \leq d-1$.

Step 4. Combining the inequalities in Step 2 and 3, we get $||D_{te_i}(\nabla u)||_{L^2} \leq ||f||_{L^2}$ for $1 \leq j \leq d-1$. Finally using Step 1, we get $\frac{\partial^2 u}{\partial x_i \partial x_k}$ exists in $L^2(\mathbb{R}^d_+)$ for all $1 \leq i \leq d-1$ and $1 \leq k \leq d$.

Step 5. It remains to prove that $\dfrac{\partial^2 u}{\partial x_d^2}$ exists in $L^2(\mathbb{R}^d_+)$. This follows from the equation since

$$\frac{\partial^2 u}{\partial x_d^2} = -\sum_{j=1}^{d-1} \frac{\partial^2 u}{\partial x_j^2} + u - f \in L^2(\mathbb{R}^d_+).$$

4. Case III: Ω is a bounded open set with C^2 boundary Use partition of unity for the boundary of the domain Ω. This results in an equation for the interior of Ω. This equation holds in all of $\mathbb{R}^d$. Hence, Case I applies. One also obtains equations near the boundary which can be mapped through a change of variables to a domain similar to the upper half plane. The proof of regularity in this case is similar to the case of the upper half plane. The transformed equation is of the form (4). For details, see H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer.

5. The method of difference quotients for proving regularity theorems is due to L. Nirenberg and is called as the *method of translations*. The importance of the method lies in the fact that it can be used to prove regularity theorems for variable-coefficient partial differential equations. For example, Let $A = (a_{jk}(x))_{j,k=1}^d$ be a matrix with Lipschitz continuous entries in Ω satisfying the ellipticity condition, i.e., there exists $\alpha > 0$ such that $\sum_{j,k=1}^d a_{jk}(x)\xi_j\xi_k \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and almost everywhere $x \in \Omega$, then the equation

$$\begin{cases} -\sum_{j,k=1}^d \frac{\partial}{\partial x_k}\left(a_{jk}(x)\frac{\partial u}{\partial x_j}\right) + u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{4}$$

has a solution $u \in H_0^1(\Omega)$ by Lax-Milgram lemma for $f \in L^2(\Omega)$. However, by the method of translations, we can prove that $u \in H^2(\Omega)$.

Some Problems

1. Let $u \in \cap_{m=0}^{\infty} H^m(\Omega)$, then prove that $u \in C^\infty(\Omega)$. Further, if Ω is a bounded open set with C^1 boundary then prove that $u \in C^\infty(\overline{\Omega})$

2. The regularity theorem tells us that $u \in H^{m+2}(\Omega)$ if $f \in H^m(\Omega)$ where u solves $-\Delta u + u = f$ in Ω with zero Dirichlet boundary conditions. Use the embedding theorems to find an integer m such that if $f \in H^m(\Omega)$ then u is a classical solution of the equation.

3. Show that the function $u(x,y) = (x^2 - y^2)\log(x^2 + y^2)$ defined in $\mathbb{R}^2$ is locally bounded and satisfies $-\Delta u = f$ in the sense of distributions for a certain $f \in L^\infty(\mathbb{R}^2)$ but is not in $W^{2,\infty}_{\text{loc}} = C^{1,1}_{\text{loc}}(\mathbb{R}^2)$.

4. Find $s \in \mathbb{R}$ such that the Dirac delta distribution δ_0 belongs to $H^s(\mathbb{R}^d)$.

5. Without using embedding theorems, prove that for $1 < p \leq \infty$, $W^{1,p}(0,1)$ is continuously embedded in $C^{0,1-1/p}$. (Hint: Absolutely continuous functions satisfy fundamental theorem of calculus.)

6. Let p be such that $1 < p \leq \infty$. Let (u_m) be a bounded sequence in $W^{1,p}(0,1)$. Show that there is a subsequence (u_{m_k}) such that (u_{m_k}) converges in $L^\infty(0,1)$. Construct a counterexample for the case of $p = 1$.

7. Let T be a distribution in $\mathbb{R}^d$ with the property that $\langle T, \phi\rangle \geq 0$ for all non-negative $\phi \in \mathcal{D}(\mathcal{R}^d)$. Show that u is of order 0.