# National Programme on Differential Equations: Theory, Computation and Applications

## Summer Internship Report

**Wigner Transforms for Wigner Measures**

*Author:*

Vivek Tewary

*Supervisor:*

Prof. Sivaji Ganesh Sista

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## Certificate

This is to certify that Vivek Tewary has attended and completed the Summer Internship organized under the National Programme on Differential Equations at IIT Bombay.

*Author:*

Vivek Tewary

*Supervisor:*

Prof. Sivaji Ganesh Sista

# Contents



**Abstract**

The aim of this report is to introduce Wigner Transforms as a stepping stone to investigating their role in Quantum Mechanics and its classical limits. We would, at some point, also like to introduce the physical and the mathematical motivation for the particular form that Wigner Transform takes. Finally, Wigner Measures will be presented.

# 1 Introduction

Quantum Mechanics prides itself on describing nature at all levels. Its peculiarities were, however, only manifest at subatomic levels until modern technological advancements had brought them to our homes. Quantum mechanics has the distinction of not only producing classical mechanics in a limiting way but also being dependent on classical mechanics for its very formulation. Wigner Transforms illustrate this point very nicely as we shall see in this report.

In Quantum Mechanics, the *state* of a particle is *completely* described by its *wave function*[1], $\psi$, which is a complex-valued function of time and space coordinates $x \in \mathbb{R}^d$. The physical attributes of a particle may only be known in a probabilistic manner, with help from the wave function. This probabilistic interpretation stems from the Heisenberg Uncertainty Principle (HUP). This is another place where classical mechanics makes its presence known in QM. For people aware of the Hamiltonian formulation of classical mechanics, HUP states that two *canonically conjugate* variables cannot be measured simultaneously to arbitrary degree of accuracy. For example the $i-th$ component of position and $i-th$ component of momentum are two such quantities.

In the mathematical formalism, the wave function is a unit element in a Hilbert space, usually $L^2(\mathbb{R}^d)$ and to every physical quantity, $a(x,p)$, for example, position, momentum etc., is associated a self-adjoint unbounded operator $a^W(x,D)$. The average value (called the *expectation value*) of an operator is given by

$$< a^W(x,D) >= \int_{\mathbb{R}^d} \bar{\psi}(x) a^W(x,D) \psi(x) dx.$$

Further, $|\psi(x)|^2$ is interpreted as the probability distribution of finding a particle in a region in space.

The time evolution of the wave function is given by the Schrödinger equation(SE)

$$i\epsilon \frac{\partial}{\partial t}\psi^\epsilon(x,t) = -\frac{\epsilon^2}{2}\Delta\psi^\epsilon(x,t) + V(x)\psi^\epsilon(x,t); \;\; \psi^\epsilon(x,0) = \psi_0^\epsilon(x) \in L^2(\mathbb{R}^d).$$

Here, $\epsilon$ plays the role of Planck's constant.[2] For $V \in C^\infty(\mathbb{R}^d)$, bounded below, the Hamiltonian operator $H^\epsilon = -\frac{\epsilon^2}{2}\Delta + V$, initially defined on $C_c^\infty(\mathbb{R}^d)$, is essentially self-adjoint in $L^2(\mathbb{R}^d)$. This assures us of solution to the SE through the spectral theorem for unbounded self-adjoint operators[RS81], [RS75]. Further, it leads to such results as conservation of energy and mass etc. through the unitarity of the evolution semigroup.

[1]Strictly speaking its *pure state* is described by a wave function. A *mixed state* requires a density operator.

[2]It is understood that all physical variables have been rescaled so that only one dimensionless quantity $\epsilon$ remains.

## 1.1 Classical Limits

It is taken as a general principle in Quantum Mechanics that Classical Mechanics should be a limiting case of Quantum Mechanics as the Planck's constant, $h$ goes to zero. Planck's constant is, of course, a constant and by its vanishing limit we mean the vanishing limit of some dimensionless parameter obtained after appropriately scaling the Schrödinger equation. Should we analyze the limit of the wave function $\psi^\epsilon$? Since $\psi^\epsilon \in L^2(\mathbb{R}^d)$, should we ask for strong convergence in $L^2$ to some limiting function which solves some classical equation? This is, in fact, a futile hope. Solutions of partial differential equations like the Schrödinger equation show such phenomena as **oscillations** and **concentrations** which we do not define but indicate through some examples and pictures.

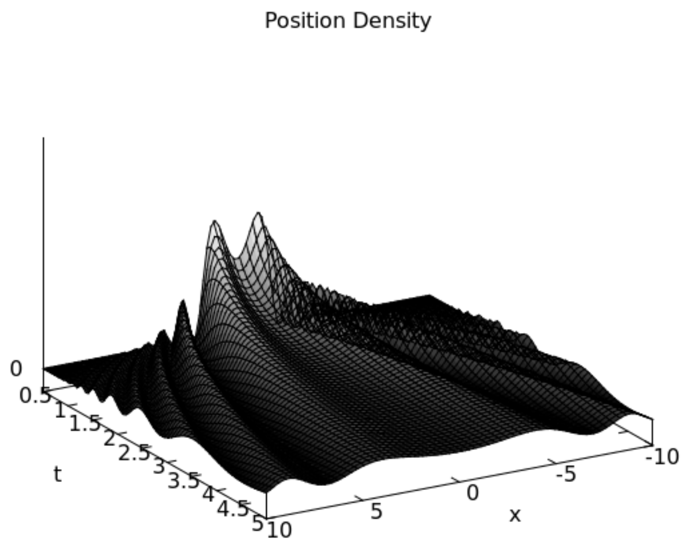


Figure 1: Evolution of position density for a free particle

Oscillations and Concentrations prevent functions from having a strong limit. This is evident from the example of $f_n(x) = sin(nx)$ in $L^2([0, 2\pi])$ where strong convergence fails but weak convergence holds. A corresponding example for concentrations can be constructed by taking $\rho \in L^2(\mathbb{R}^d)$, $supp\ \rho \subseteq \overline{B}(0,1)$, $\int \rho^2 dx = 1$ and defining $\rho^\epsilon(x) = \epsilon^{-d/2}\rho(x/\epsilon)$. This converges weakly to 0 in $L^2(\mathbb{R}^d)$.

We may compute weak limit of the wave functions or weak-* limit in the space of measures.

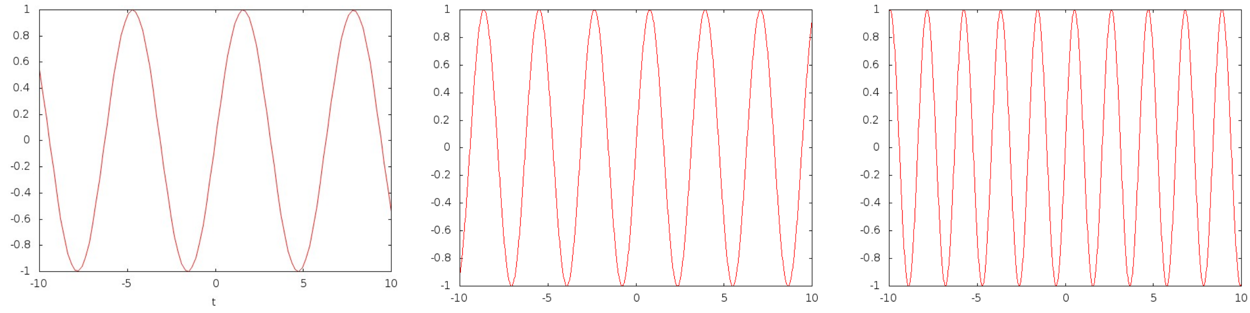


Figure 2: Oscillations

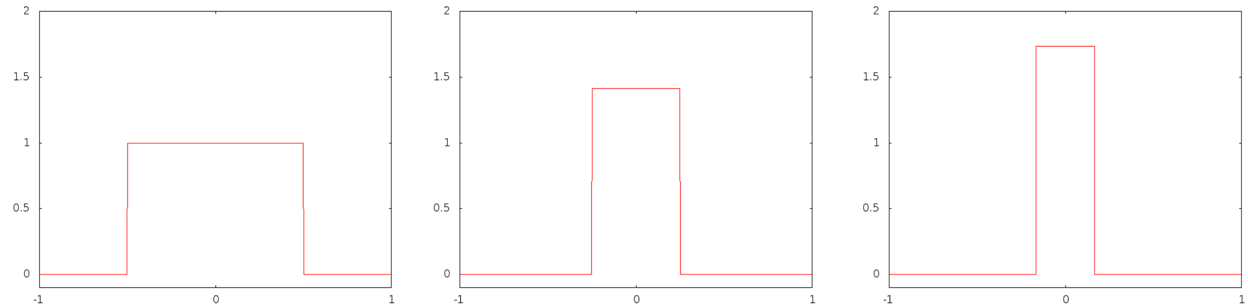

Figure 3: Concentrations

These however will not allow us to compute limits of expectation values of observables because these are quadratic functions of wave functions and it is a well known fact that weak limits do not commute with nonlinear operations. As example again consider $f_n(x) = sin(nx)$ with weak limit 0 in $L^2([0, 2\pi])$, whereas, $f_n^2 \rightharpoonup \frac{1}{2}$.

## 2 Wigner Transforms

At this point, Wigner Transforms come to the rescue. In 1932, Eugene Wigner[Wig32] came up with an equivalent phase-space formulation of Quantum Mechanics in terms of, what are now called, Wigner Transforms. For $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$w^\epsilon(f,g)(x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f\left(x - \frac{\epsilon v}{2}\right) \bar{g}\left(x + \frac{\epsilon v}{2}\right) e^{i\xi \cdot v} dv.$$

To begin with, we would like to define the Wigner Transform for more general class of functions. For $F \in \mathcal{S}(\mathbb{R}^{2d})$, we would like to define two operations

$$T_s^\epsilon F(x,v) = F\left(x - \frac{\epsilon v}{2}, x + \frac{\epsilon v}{2}\right).$$

$$\tilde{T}_s^\epsilon F(x,v) = \frac{1}{\epsilon^d} F\left(\frac{x+v}{2}, \frac{v-x}{\epsilon}\right).$$

For $F \in \mathcal{S}(\mathbb{R}^{2d})$, the Fourier Transform in the second variable is defined as

$$\mathcal{F}_2 F(x,v) = \int_{\mathbb{R}^d} F(x,\xi) e^{-iv\cdot\xi} d\xi.$$

and the inverse Fourier transform in the second variable is defined as

$$\tilde{\mathcal{F}}_2 F(x,v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} F(x,\xi) e^{iv\cdot\xi} d\xi.$$

These four definitions we would like to extend to more general functions spaces, in particular, tempered distributions are most suitable for this purpose.

For $f \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\phi \in \mathcal{S}(\mathbb{R}^{2d})$,

$$< T_s^\epsilon f, \phi > = < f, \tilde{T}_s^\epsilon \phi >$$

$$< \tilde{T}_s^\epsilon f, \phi > = < f, T_s^\epsilon \phi > .$$

$$< \mathcal{F}_2 f, \phi > = (2\pi)^d < f, \tilde{\mathcal{F}}_2 \phi >$$

$$< \tilde{\mathcal{F}}_2 f, \phi > = \frac{1}{(2\pi)^d} < f, \mathcal{F}_2 \phi > .$$

These definitions are motivated by duality and hold in the usual sense when the distributions come from Schwartz class functions, for example. We stress that since these are distributions acting on complex valued functions and we make the convention that the distribution acts on the complex conjugate of the function; This allows us to match up the action of a distribution with the $L^2$ inner product in appropriate situations.

These then also allow us to give a more general definition for Wigner Transforms. For $f, g \in \mathcal{S}'(\mathbb{R}^d)$

$$w^\epsilon(f,g) = \tilde{\mathcal{F}}_2 T_s^\epsilon (f\bar{g}).$$

We shall get back to the matter of Wigner measures after a detour through Weyl Quantization.

## 3 Weyl Quantization

For physically relevant reasons, the operator associated with the physical quantity *position* is multiplication by the position coordinate and the operator associated with the $i-th$ component of momentum is $\frac{\epsilon}{i}\frac{\partial}{\partial x_i}$. These considerations led Weyl[Wey27] to a quantization scheme for classical physical observables (smooth functions on the phase space $\mathbb{R}^d_x \times \mathbb{R}^d_p$), $a(x,p)$. The proper context in which to study Weyl Quantization is the theory of pseudodifferential operators and microlocal analysis. For our purposes, it will suffice to define Weyl Quantization in simpler situations involving Schwartz Class, $\mathcal{S}(\mathbb{R}^d)$, the class of tempered distributions, $\mathcal{S}'(\mathbb{R}^d)$ and at most, $L^2(\mathbb{R}^d)$. A study of pseudodifferential operators is a task for another time.

For $a \in \mathcal{S}(\mathbb{R}^{2d})$ and $f \in \mathcal{S}(\mathbb{R}^d)$,

$$a^W(x,\epsilon D)f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2},\epsilon\xi\right) f(y)\, e^{i(x-y)\cdot\xi} d\xi dy$$

is well-defined and the "operator" $a^W(x,\epsilon D)$ is known as the Weyl-Quantization of the "symbol" $a$. In the perfectly smooth setting the above may be seen as an integral operator with kernel given by

$$K^\epsilon(x,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} a\left(\frac{x+y}{2},\epsilon\xi\right) e^{i(x-y)\cdot\xi} d\xi.$$

This expression can be written in a form that allows for extension to more general spaces.

$$K^\epsilon = \frac{1}{(2\pi)^d}\tilde{T}^\epsilon_s \mathcal{F}_2 a.$$

We see $a^W(x,\epsilon D)f$ as a tempered distribution that acts on $g \in \mathcal{S}(\mathbb{R}^d)$ through the following equation

$$< a^W(x,\epsilon D)f, g > := < K^\epsilon, g\overline{f} > .$$

The above would make sense even if $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $f, g \in \mathcal{S}(\mathbb{R}^{2d})$. The great advantage of this equation is that it allows us to connect Weyl Quantization with Wigner Transforms.

$$\begin{aligned} < a^W(x,\epsilon D)f,g > &= < K^\epsilon, g\overline{f} > \\ &= \frac{1}{(2\pi)^d} < \tilde{T}_s^\epsilon \mathcal{F}_2 a, g\overline{f} > \\ &= \frac{1}{(2\pi)^d} < \mathcal{F}_2 a, T_s^\epsilon(g\overline{f}) > \\ &= < a, \tilde{\mathcal{F}}_2 T_s^\epsilon(g\overline{f}) > \\ &= < a, w^\epsilon(g,f) > \end{aligned}$$

# 4 Wigner Measures

As mentioned earlier, for $f, g \in \mathcal{S}^{'}(\mathbb{R}^d)$, Wigner Transforms enjoy the more general definition

$$w^\epsilon(f,g) = \tilde{\mathcal{F}}_2 T_s^\epsilon(f\bar{g}).$$

Suppose that $f, g$ come from a bounded set in $L^2(\mathbb{R}^d)$, then this definition makes it clear that the Wigner Transforms $\{w^\epsilon(f,g)\}$ will also lie in a bounded set of $L^2(\mathbb{R}^{2d})$ independent of $\epsilon$. However, for our purposes, it suffices that the Wigner Transform lies in bounded subset of $\mathcal{S}^{'}(\mathbb{R}^{2d})$ independent of $\epsilon$. Now, Banach Alaoglu Theorem[Rud91] allows us to obtain a subsequence which converges weak-$*$ to a tempered distribution $w$ (the choice of subsequence is by no means, unique). The next estimate makes it possible to prove that the limiting distribution is a positive measure.

**Proposition 1** *(Gérard et al. 1997[GMMP97]) For $a, b \in \mathcal{S}(\mathbb{R}^{2d})$,*

$$< w^\epsilon(f,g), a\overline{b} >= \int_{\mathbb{R}^d} (a^W(x,\epsilon D_x)f)(\overline{b^W(x,\epsilon D_x)g})dx + r_\epsilon$$

*where $|r_\epsilon| \leq C(a,b)||f||||g||$*

First, observe that when $a \in \mathcal{S}(\mathbb{R}^{2d})$, the "operator" $a^W(x,\epsilon D)$ takes $\mathcal{S}^{'}$ into $\mathcal{S}$. The proof requires the identity

$$< w^\epsilon(f,g), a\overline{b} >= \int f\left(x - \epsilon\frac{v}{2}\right)\overline{g}\left(x + \epsilon\frac{v}{2}\right)\tilde{\mathcal{F}}_2 a(x,u)\overline{\tilde{\mathcal{F}}_2} b(x,u-v)dxdudv.$$

To see this is easy once we know how to write the product of two inverse Fourier transforms as a convolution. Now make a change of variable, $v = u - u^{'}$ and $x = x^{'} - \epsilon\frac{u+u^{'}}{2}$. The identity becomes

$$< w^{\epsilon}(f,g), a\bar{b} >= \int f(x^{'} - \epsilon u)\tilde{\mathcal{F}}_2 a\left(x^{'} - \epsilon \frac{u+u^{'}}{2}, u\right) \bar{g}(x^{'} - \epsilon u^{'})\overline{\tilde{\mathcal{F}}_2 b}\left(x^{'} - \epsilon \frac{u+u^{'}}{2}, u^{'}\right) dx^{'} du du^{'}.$$

The expressions in $a, b$ are ready-made to apply the Fundamental Theorem of Calculus. Appropriate manipulations then yield the required estimate.

We can show $w(f,f)$, the limit of the Wigner Transforms for $f = g$, is a measure by taking $f = g$ in the above estimate and noting that every nonnegative function $c \in C_c^{\infty}(\mathbb{R}^d)$ is obtainable as the limit of $|d_n|^2$ for an appropriate sequence $\{b_n\}$ in $C_c^{\infty}(\mathbb{R}^d)$. Therefore, this limit is known as a Wigner Measure.

Now, we must elucidate the relation of Wigner Measures to the semiclassical limits of Quantum Mechanics. As mentioned earlier, the solution to the Schrödinger equation exist under fairly mild conditions on the potential $V$, for example, $V$ should be $C^1$ and bounded below. These solutions stay in $L^2(\mathbb{R}^d)$ for all time due to the unitarity of the evolution semigroup. The Wigner Transform of the solution of the Schrödinger equation satisfies the Wigner equation namely

$$\frac{\partial}{\partial t} w^{\epsilon} + p \cdot \nabla_x w^{\epsilon} + \Theta^{\epsilon}[V] w^{\epsilon} = 0, \;\; w^{\epsilon}|_{t=0} = w_0^{\epsilon}$$

where

$$(\Theta^{\epsilon}[V]f)(x,p) := -\frac{i}{(2\pi)^d} \int \int_{\mathbb{R}^d} \frac{1}{\epsilon} \left( V\left(x + \frac{\epsilon}{2} y\right) - V\left(x - \frac{\epsilon}{2} y\right)\right) f(x,q) e^{iy.(p-q)} dy dq.$$

Notice that this equation closely resembles the Liouville equation from Classical Mechanics. In fact, formally, the last term goes to $\nabla_x V \cdot \nabla_p w^0$ in the vanishing limit of $\epsilon$. This can be made rigorous as the following theorem suggests.

**Theorem 2** *(Lions, Paul '93[PL93]) Suppose V is such that the Hamiltonian in the Schrödinger equation is essentially self-adjoint. For definiteness, let* $V \in C^{\infty}(\mathbb{R}^d)$ *and* $V \geq 0$.

1. *The limiting Wigner measure corresponding to the solution of the Schrödinger equation satisfies in the sense of distributions the following equation*

$$\frac{\partial}{\partial t} w_t^0 + p \cdot \nabla_x w_t^0 - \nabla_x V \cdot \nabla_p w_t^0 = 0, \;\; w^0|_{t=0} = w_0^0$$

2. *If, in addition, $V \in C^{1,1}(\mathbb{R}^d)$, then $w^0$ is the unique solution of the above equation in $C_b(\mathbb{R}, \mathcal{M}_+(\mathbb{R}^{2d}))$. Further, the Wigner Measure is transported along the Hamiltonian Flow associated with the differential equation*

$$\begin{aligned} \dot{x} &= p \\ \dot{p} &= -\nabla V(x). \end{aligned}$$

*In other words, if $\Phi_t$ is the Hamiltonian flow of the above system of differential equations, then the Wigner Measure at time $t$ is the push-forward under this flow map of the Wigner Measure at time $t = 0$. For $f$ measurable,*

$$\int\int_{\mathbb{R}^{2d}} f(x,p)dw^0(x,p,t) = \int\int_{\mathbb{R}^{2d}} f \circ \Phi_t(x,p)dw^0(x,p,0).$$

## 4.1 Examples

- For **concentrating functions**, i.e., $\psi^\epsilon(x) = (\epsilon)^{-d/2} f\left(\frac{x-x_0}{\epsilon}\right)$ for $f$ smooth and compactly supported, we have $w^\epsilon(x,p) = \frac{1}{\epsilon^d} w\left(\frac{x-x_0}{\epsilon}, p\right)$ where $w(x,p) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f\left(x+\frac{y}{2}\right)\overline{f}\left(x-\frac{y}{2}\right)e^{iy\cdot p}dy$. Observe that

$$\begin{aligned} < w^\epsilon, \phi > &= \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_p} \phi(x,p)\frac{1}{\epsilon^d} w\left(\frac{x-x_0}{\epsilon}, p\right) dpdx \\ &= \int_{\mathbb{R}^d_p}\left\{\int_{\mathbb{R}^d}(\phi(x_0+\epsilon z,p)-\phi(x_0,p))w(z,p)dz + \phi(x_0)\int_{\mathbb{R}^d} w(z,p)dz\right\}dp \\ &\to \int_{\mathbb{R}^d_p}\int_{\mathbb{R}^d}\phi(x_0,p)w(z,p)dzdp \\ &= \int_{\mathbb{R}^d_p}\int_{\mathbb{R}^d_x}\phi(x,p)dw(x,p) \end{aligned}$$

  where

$$w(x,p) = \delta_{x_0} \otimes \int_{\mathbb{R}^d} w(z,p)dz = (2\pi)^{-d}|\widehat{f}(p)|^2\delta_{x_0}.$$

  This last equality comes from the fact that the $x$-moment of Wigner Transform is the modulus squared of Fourier Transform of the function $f$. (and similarly, the $p$-moment of Wigner Transform is the modulus squared of the function.) These can be easily verified from the formula and the usual Fourier identities.

- For **semiclassical wave packets**, i.e., $\psi^\epsilon(x) = \epsilon^{-d/4} f\left(\frac{x-x_0}{\epsilon}\right) e^{ip_0\cdot x/\epsilon}$, after some calculation the Wigner Measure is found to be

$$w(x,p) = \left(\int_{\mathbb{R}^d} |f(x)|^2 dx\right) \delta_{x_0} \otimes \delta_{p_0}.$$

These examples are adapted from the paper of Lions and Paul[PL93].

# A Fourier Transform

We define the **Schwartz class,** $\mathcal{S}$ of functions to be the space

$$\mathcal{S}(\mathbb{R}^d) = \left\{ \phi \in C^\infty(\mathbb{R}^d;\mathbb{C}) : \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta \phi(x)| < \infty \textit{ for all } \alpha,\beta \in \mathbb{N}^d \cup \{0\} \right\}.$$

The function that, in some sense, characterizes the Schwartz class is $\phi(x) = exp\left(-\frac{1}{2}|x|^2\right)$. The theory of Fourier transform is progresses in the easiest manner for the Schwartz class. We define the Fourier transform for $\phi \in \mathcal{S}(\mathbb{R}^d)$ as

$$\hat{\phi}(\xi) = \mathcal{F}(\phi)(\xi) = \int_{\mathbb{R}^d} \phi(x) e^{-ix\cdot\xi} dx.$$

Indeed this integral makes sense for $\phi \in L^1(\mathbb{R}^d)$ and consequently for $\phi \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$. This appendix is to serve as a quick reminder of the important properties of Fourier Transform. Proofs may be found in books such as Kesavan[Kes89], Zworski[Zwo12].

**Theorem 3** *1. Fourier Transform maps $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$.*

*2.*

$$\mathcal{F}(exp\left(-\frac{1}{2}|x|^2\right)) = (2\pi)^{d/2} exp\left(-\frac{1}{2}|\xi|^2\right).$$

*3. Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$ isomorphically. The inverse transform is defined by*

$$\mathcal{F}^{-1}(\psi)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(\xi) e^{ix\cdot\xi} d\xi$$

*so that*

$$\phi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) e^{ix\cdot\xi} d\xi.$$

*4.*

$$\frac{1}{(i)^{|\alpha|}} D^\alpha_\xi(\mathcal{F}(\phi)) = \mathcal{F}((-x)^\alpha \phi)$$

*5.*

$$\mathcal{F}\left(\frac{1}{(i)^{|\alpha|}} D_x^\alpha \phi\right) = \xi^\alpha \mathcal{F}(\phi)$$

*For $\psi, \phi \in \mathcal{S}(\mathbb{R}^d)$,*

*6.*

$$\int_{\mathbb{R}^d} \hat{\phi}\psi dx = \int_{\mathbb{R}^d} \phi\hat{\psi} d\xi.$$

*7.*

$$\int_{\mathbb{R}^d} \phi\overline{\psi} dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}\overline{\hat{\psi}} d\xi.$$

*so that*

$$||\psi||^2_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^d} ||\hat{\psi}||^2_{L^2(\mathbb{R}^d)}.$$

In particular, the last equality allows us to extend the Fourier transform upto $L^2(\mathbb{R}^d)$ as an isometry. In that case, The Fourier transform can no longer be thought of as an integral but as a limits of integrals. Moreover, the Fourier Inversion formula holds also in the case when both $\phi$ and $\hat{\phi}$ belong to $L^1(\mathbb{R}^d)$ and then, $\phi$ agrees with a continuous function almost everywhere. In case, $\phi \in L^1(\mathbb{R}^d)$, the Fourier inversion formula can be made sense of in the following sense, for Lebesgue almost everywhere $x \in \mathbb{R}^d$,

$$\phi(x) = \lim_{\epsilon \to 0} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-\epsilon^2\xi^2/2} \hat{f}(\xi) d\xi.$$

Some mention need also be made of tempered distributions. Suffice to say that they are the topological dual space of the Schwartz class, $\mathcal{S}$. The Schwartz class is a Frechét space with countably many semi-norms given by

$$p_N(\phi) = \sup_{|\alpha|,|\beta| \le N} |x^\alpha D^\beta \phi(x)|.$$

Tempered distributions find an important place in Fourier Analysis and the theory of Distributions[1]. Note that since we are talking about complex valued functions, tempered distributions act on complex conjugate of Schwartz class functions. This convention has the advantage that it corresponds to the $L^2$ inner product when the need arises. The definition of Fourier transform may be extended to tempered distributions in the following manner. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, then $< \mathcal{F}(f), \phi >:= (2\pi)^d < f, \mathcal{F}^{-1}(\phi) >$ and $< \mathcal{F}^{-1}(f), \phi >:= \frac{1}{(2\pi)^d} < f, \mathcal{F}(\phi) >$.

[1] invented by Laurent Schwartz for which he receieved the Fields Medal in 1950.

## B Notation

| | |
|---|---|
| $\mathcal{M}^+(\mathbb{R}^d)$ | Space of positive measures on $\mathbb{R}^d$ |
| $C_0(\mathbb{R}^d)$ | Completion of the space of continuous functions on $\mathbb{R}^d$ with compact support under supremum norm |
| $D^\alpha$ | $\frac{\partial^{\alpha_1}}{\partial x_1}\frac{\partial^{\alpha_2}}{\partial x_2}\dots\frac{\partial^{\alpha_d}}{\partial x_d}$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ |
| $\lvert\alpha\rvert$ | $\lvert\alpha_1\rvert + \lvert\alpha_2\rvert + \dots + \lvert\alpha_d\rvert$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ |
| $x^\alpha$ | $x_1^{\alpha_1}x_2^{\alpha_2}\dots x_d^{\alpha_d}$ for $x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d \cup \{0\}$ |


## C Acknowledgements

This is to acknowledge that the formulation of Wigner Transforms and Weyl Quantization in terms of operations like $T_s^\epsilon$ and $\mathcal{F}_2$ is inspired and motivated from the books of Gröchenig[Grö01] and Folland[Fol89].


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