Generic Simplicity of Spectral Edges and Bloch Wave Homogenization of Almost Periodic Media

A Thesis

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

by

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Abstract

This thesis is devoted to the Bloch wave method which is a spectral method in the theory of homogenization. Homogenization theory can be traced to the study of composite materials, where the equations contain a small parameter representing the scale of heterogeneities in the material. The Bloch wave method was developed by Conca and Vanninathan. It relies on the direct integral decomposition of periodic elliptic operators. The spectrum of such operators is a union of intervals and spectral edges determine a variety of physical phenomena. Homogenization may be thought of as one such spectral edge (or threshold) effect. As such, the Bloch wave method relies on regularity properties of the spectral edges, i.e., smoothness of Bloch eigenvalues close to a spectral edge and structural properties of the coefficients, i.e., periodicity.

The first part of this thesis consists of studying the regularity properties of spectral edges. Generic simplicity of spectral edges under various regularity assumptions on the coefficients is proved. In particular, a perturbation of the coefficients results in a spectral edge which is attained by only one Bloch eigenvalue. It must be recalled that for the applications to homogenization, it is necessary to consider coefficients that are only measurable and bounded. By employing such a result, certain resolvent estimates for internal edge homogenization are established for a multiple spectral edge.

The second part of this thesis extends the Bloch wave method to a class of media not satisfying the structural property of periodicity. This task is performed for almost periodic media through periodic approximations on cubes of increasing size. On the way, an interesting module containment result is proved. A rate of convergence result for approximate homogenized tensors is also proved under a decay condition on a modulus of almost periodicity. This decay condition subsumes the well-known Kozlov condition for quasiperiodic coefficients. Finally, Bloch wave homogenization is achieved for quasiperiodic operators by lifting to degenerate periodic operators.

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$\mathbb{R}$	The space of all real numbers.
$\mathbb{N}$	The set of all natural numbers.
$\mathbb{Z}$	The set of all integers.
$\mathbb{Q}$	The set of all rational numbers.
$\mathbb{C}$	The complex field.
K	K will be used to denote either the fields $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{R}^d$	d-dimensional Euclidean space.
$\mathbb{C}^d$	d-tuples with complex entries.
$\mathbb{T}^d$	d-dimensional flat torus.
$C_c^\infty(\mathbb{R}^d)$	The space of all compactly supported smooth functions on $\mathbb{R}^d$. With the topology of test functions, it is denoted by $\mathcal{D}(\mathbb{R}^d)$.
$L^p(\mathbb{R}^d)$	The Banach space of all square-integrable functions on $\mathbb{R}^d$ where p is such that $1 \leq p < \infty$.
$L^\infty(\mathbb{R}^d)$	The Banach space of all measurable and essentially bounded functions on $\mathbb{R}^d$.
Ω	A domain in $\mathbb{R}^d$.
$L^p(\Omega)$	The Banach space of all p-integrable functions on Ω where p is such that $1 \leq p < \infty$.
$L^\infty(\Omega)$	The Banach space of all measurable and essentially bounded functions on Ω.
$H^1(\Omega)$	The Hilbert space of all square integrable functions on Ω whose weak derivatives are also square integrable.
$H_0^1(\Omega)$	The subspace of $H^1(\Omega)$ whose elements have zero trace on $\partial\Omega$.
Y	The set $[0, 2\pi)^d$, also the 2π Periodicity cell in $\mathbb{R}^d$.
Y'	The set $\left[-\frac{1}{2}, \frac{1}{2}\right)^d$, also the dual periodicity cell in $\mathbb{R}^d$.
T_p	Translation by $2\pi p$ for $p \in \mathbb{Z}^d$, i.e., $T_p f(y) = f(y + 2\pi p)$

$C^\infty_\sharp(Y)$	The space of all smooth Y-periodic functions on $\mathbb{R}^d$.
$C^\infty_\sharp(\eta, Y)$	$e^{i\eta\cdot y}C^\infty_\sharp(Y)$
$L^p_\sharp(Y)$, $L^p(\mathbb{T}^d)$	The closure of $C^\infty_\sharp(Y)$ in the norm of $L^p(Y)$ where p is such that $1 \le p < \infty$.
$H^1_\sharp(Y)$, $H^1(\mathbb{T}^d)$	The closure of $C^\infty_\sharp(Y)$ in the norm of $H^1(Y)$.
$L^\infty_\sharp(Y)$	The space of all equivalence classes of functions f in $L^\infty(\mathbb{R}^d)$ such that $T_p f(x) = f(x)$ a.e. $x \in \mathbb{R}^d$ for all $p \in \mathbb{Z}^d$.
$L^p_\sharp(\eta, Y)$	The closure of $C^\infty_\sharp(\eta, Y)$ in the norm of $L^p(Y)$ where p is such that $1 \le p < \infty$.
$H^1_\sharp(\eta, Y)$	The closure of $C^\infty_\sharp(\eta, Y)$ in the norm of $H^1(Y)$.
$L^2(Y'; L^2_\sharp(Y))$	The Bochner space of $L^2_\sharp(Y)$-valued functions on Y' with finite $L^2(Y')$ norm.
$L^2(Y'; \ell^2(\mathbb{N}))$	The Bochner space of all $\ell^2(\mathbb{N})$-valued functions on Y' with finite $L^2(Y')$ norm.
A	A will denote a matrix whose entries are $L^\infty(\mathbb{R}^d)$ functions. A will be positive definite. A will either be periodic or almost periodic.
Sym(d)	The set of all symmetric $d \times d$ matrices with real entries.
$M^>_B$	$\{A : \mathbb{R}^d \to Sym(d) : a_{kl} \in L^\infty_\sharp(Y, \mathbb{R})$ and A is coercive$\}$
$\mathcal{A}$	$\mathcal{A}$ denotes the operator $-\nabla \cdot (A\nabla)$
$\mathcal{A}(\eta)$	$\mathcal{A}(\eta)$ denotes the shifted operator $-(\nabla + i\eta)\cdot(A(\nabla + i\eta))$.
$\int^\oplus_{Y'} \mathcal{A}(\eta)\, d\eta$	Direct integral of $\mathcal{A}(\eta)$ over Y'.
$\sigma(\mathcal{A})$	Spectrum of $\mathcal{A}$.
$R(\zeta, \mathcal{A})$	Resolvent of operator $\mathcal{A}$ at ζ, i.e., $(\mathcal{A} - \zeta I)^{-1}$.
$\mathcal{M}_Y$	The mean value of periodic functions over periodicity cell Y.
Y_L	Y_L denotes the set $[-\pi L, \pi L)^d$ for $L > 0$.
$\mathcal{M}$	The mean value of $L^1_{loc}(\mathbb{R}^d)$ functions defined as $\mathcal{M}(u) = \limsup_{T\to\infty} \frac{1}{\lvert Y_T\rvert}\int_{Y_T} u(y)\, dy$.
$Trig(\mathbb{R}^d; K)$	The space of all K-valued trigonometric polynomials of the form $P(y) = \sum_{j=1}^{N} a_j e^{iy\cdot\eta_j}$, defined for $y \in \mathbb{R}^d$.

$AP(\mathbb{R}^d)$	The space of Bohr almost periodic functions, i.e., the closure of $\mathrm{Trig}(\mathbb{R}^d)$ in the uniform norm.
$B^p(\mathbb{R}^d)$	The space of Besicovitch almost periodic functions, i.e., the closure of $\mathrm{Trig}(\mathbb{R}^d)$ in the seminorm $\lvert u\rvert_{B^p} := \left(\lim\limits_{T\to\infty} \frac{1}{\lvert Y_T\rvert}\int_{Y_T} \lvert u(y)\rvert^p \, dy\right)^{1/p}$. The Banach space generated by quotienting the previous space by the set of functions with seminorm zero is also denoted by the same notation.
$\exp(u)$	For $u \in B^p(\mathbb{R}^d)$, the set of $\xi \in \mathbb{R}^d$ such that $\mathcal{M}(u \cdot e^{ix\cdot\xi}) \neq 0$.
$\mathrm{Mod}(u)$	The $\mathbb{Z}$-module generated by $\exp(u)$ for $u \in B^p(\mathbb{R}^d)$.

List of Publications

[1] S. Sivaji Ganesh and Vivek Tewary. Generic simplicity of spectral edges and applications to homogenization. *Asymptotic Analysis, vol. 116, no. 3-4, pp. 219-248, 2020.* https://content.iospress.com/articles/asymptotic-analysis/asy191542*; preprint at* https://arxiv.org/abs/1807.00917, 2019. Cited 4 times on pages 28, 35, 65, and 83.

[2] S. Sivaji Ganesh and Vivek Tewary. Bloch approach to almost periodic homogenization and approximations of effective coefficients. https://arxiv.org/abs/1908.07977, 2019. Accessed: 2019-08-22. Cited 3 times on pages 28, 83, and 105.

[3] S. Sivaji Ganesh and Vivek Tewary. Bloch wave homogenization of quasiperiodic media. https://arxiv.org/abs/1910.12724, 2019. Accessed: 2019-10-29. Cited 2 times on pages 28 and 131.

Chapter 1

Introduction

This thesis is concerned with asymptotic analysis of partial differential equations, specifically theory of homogenization. Qualitative homogenization theory proposes effective macroscopic equations for equations with highly oscillatory coefficients. The size of oscillations is denoted by a small parameter ϵ and the effective equation is obtained in the limit as $\epsilon \to 0$. Therefore, the qualitative theory answers the important physical question of *which models are valid at what scales*. On the other hand, the quantitative theory of homogenization aims to construct approximate solutions and establish corresponding rates of convergence, which exploits compactness methods to establish regularity results. This regularity theory relies on the fact that the homogenized tensor is a constant matrix in many cases and therefore has more regular solutions. The regularity of the limit solution is used to prove uniform regularity results for solutions at the ϵ-scale. In this chapter, we will describe the theory of homogenization, spectral theory of periodic operators, Bloch wave method of homogenization and our contributions to these areas of research.

We will study the prototype operator which exhibits all the phenomena that we wish to explore: viz. the second-order elliptic operator in divergence form, given by

$$\mathcal{A}u := -\mathrm{div}(A(y)\nabla u) = -\frac{\partial}{\partial y_k}\left(a_{kl}(y)\frac{\partial u}{\partial y_l}\right), \tag{1.1}$$

where summation convention over repeated indices is assumed. Throughout this thesis, we shall make the following assumptions on the entries of the matrix A.

(A1) The coefficients $A = (a_{kl}(y))$ are measurable bounded real-valued functions defined on $\mathbb{R}^d$. In other words, $a_{kl} \in L^\infty(\mathbb{R}^d)$.

(A2) The matrix $A = (a_{kl})$ is symmetric, i.e., $a_{kl}(y) = a_{lk}(y)$.

(A3) Further, the matrix A is *coercive*, i.e., there exists an $\alpha > 0$ such that

$$\forall \, \nu \in \mathbb{R}^d \text{ and } a.e. \, y \in \mathbb{R}^d, \langle A(y)\nu, \nu \rangle \geq \alpha \|\nu\|^2. \tag{1.2}$$

We shall additionally impose structural conditions on the coefficients of the operator. In the earlier part of the thesis, the entries of A are assumed to be periodic; whereas, in the later parts, the entries of A will be almost periodic. The notion of periodicity and almost periodicity will be defined in due course. The introduction chapter is planned as follows – In Section 1.1, we will state the central problem of homogenization theory and the main theorem in periodic homogenization. In Section 1.2, we will describe the Bloch wave method in considerable detail. In Section 1.3, we shall discuss parametrized eigenvalue problems, of which, Bloch eigenvalue problem is an example. In Section 1.4, we shall explain the notion of spectral edge of a periodic elliptic operator and discuss how it appears in Bloch wave method. In Section 1.5, we review the internal edge homogenization theorem of Birman and Suslina. In Section 1.6, we shall review the homogenization theory of almost periodic media and compare it to periodic homogenization. In Section 1.7, we will describe our contributions to the topics discussed in the previous sections and compare them to available literature.

1.1 Theory of periodic homogenization

In homogenization, one studies the limits of solutions to equations with highly oscillatory coefficients, such as

$$\mathcal{L}^\epsilon u^\epsilon = 0 \qquad \text{in } \Omega \subset \mathbb{R}^d.$$

Suppose that, as $\epsilon \to 0$, u^ϵ converges to u^* in an appropriate sense, often weakly in some function space. Then, the aim of theory of homogenization is to derive an effective operator $\mathcal{L}^*$ such that

$$\mathcal{L}^* u^* = 0 \qquad \text{in } \Omega.$$

A concrete example where such a problem arises is that of composite materials. Composite materials are obtained from a number of small-scale constituents in different configurations. Their size is much smaller compared to the global dimension of the composite. At the microscopic level, such a material is heterogeneous, whereas at a macroscopic level, the material looks homogeneous. Homogenization gives the macroscopic description of the composite material which is heterogeneous at the microscopic level.

To fix the notions, consider a model for heat conduction through a material. This is given by a partial differential equation of the form

$$\begin{cases} \mathcal{A}u = -\nabla \cdot (A\nabla u(y)) = f(y) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

where A represents the conductivity tensor for the material, u represents temperature, f is the heat source and Ω is a bounded open set in $\mathbb{R}^d$. A is a constant for a homogeneous material, however for a composite, A would be a function of $x \in \Omega$. In fact, it would be a highly oscillatory function. To simplify matters, and indeed, as is the case in many physical applications, we could assume the configuration of heterogeneities in the composite to be periodic. In later chapters of this thesis, we shall also consider almost periodic configurations.

To introduce periodicity, we define a reference cell $Y = [0, 2\pi)^d$, and cover $\mathbb{R}^d$ with translates of Y. In the reference cell, the tensor A is a function of period 2π, hereby, called, Y-periodic. The space of measurable bounded periodic real-valued functions in Y is denoted by $L^\infty_\sharp(Y, \mathbb{R})$. Hence, $a_{kl} \in L^\infty_\sharp(Y, \mathbb{R})$. In many instances, we will identify Y with a torus $\mathbb{T}^d$ and the space $L^\infty_\sharp(Y, \mathbb{R})$ with $L^\infty(\mathbb{T}^d, \mathbb{R})$, in the standard way. Let $\epsilon > 0$ be a small number much less than 1. To introduce heterogeneity in the form of arbitrarily small period, we repeat the same procedure with ϵY instead of Y. Now, the domain Ω is covered with a lattice of period ϵY. ϵY-Periodicity is introduced in the conductivity tensor by writing $A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right)$. The problem becomes

$$\begin{cases} \mathcal{A}^\epsilon u^\epsilon := -\text{div}(A^\epsilon(x)\nabla u^\epsilon(x)) = f(x) \text{ in } \Omega \\ u^\epsilon = 0 \text{ on } \partial\Omega \end{cases} \tag{1.4}$$

By a simple application of Lax-Milgram lemma, we conclude that the solutions of equation (1.4), u^ϵ are uniformly bounded in H^1 norm. Therefore, for a subsequence, $u^\epsilon \rightharpoonup u^*$ in $H^1_0(\Omega)$-weak. We shall characterize u^*, in the sense that we shall find a tensor A^* so that the following holds

$$\begin{cases} -\text{div}(A^*(y)\nabla u^*(y)) = f(y) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

The effective matrix A^* is given in terms of the solutions of the cell problem, known as correctors. All of this information is collected in the following theorem:

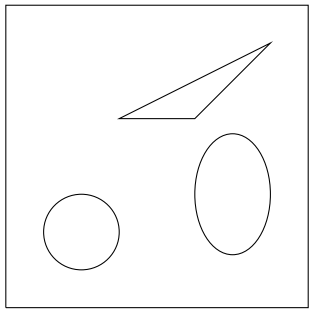


Figure 1.1: Reference cell Y

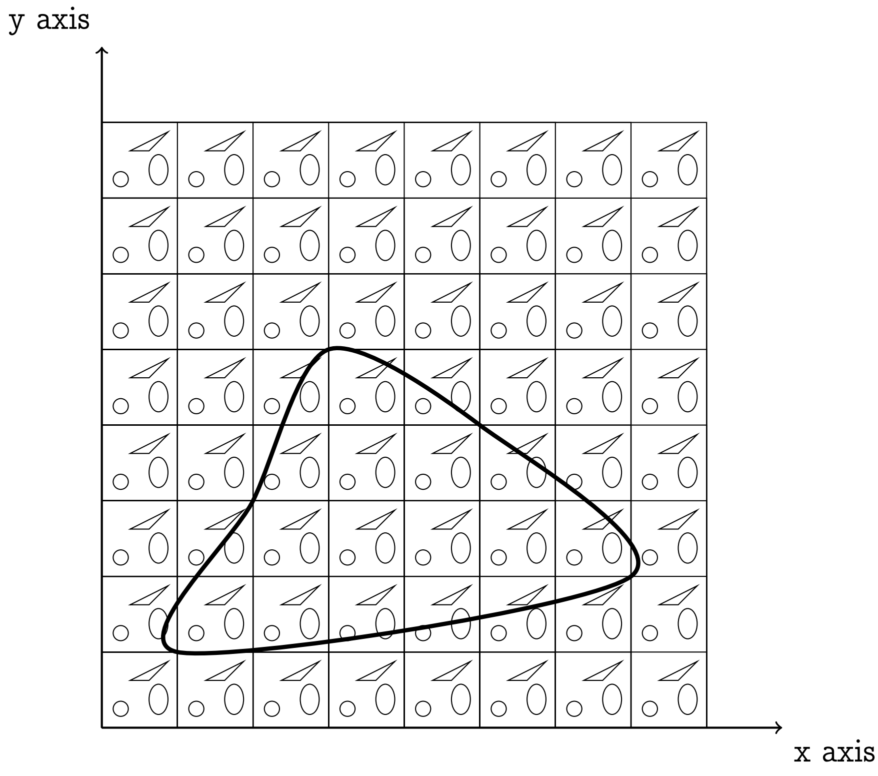


Figure 1.2: Ω in ϵY lattice

Theorem 1.1. *Let* Ω *be an arbitrary domain in* $\mathbb{R}^d$ *and* $f \in L^2(\Omega)$. *Let* $u^\epsilon \in H^1(\Omega)$ *be such that* u^ϵ *converges weakly to* u^* *in* $H^1(\Omega)$, *and*

$$-\operatorname{div}(A^\epsilon(x)\nabla u^\epsilon(x)) = f(x) \text{ in } \Omega. \tag{1.6}$$

Then, the limit u^* *satisfies the homogenized equation:*

$$\mathcal{A}^* u^* := -\operatorname{div}(A^*\nabla u^*) = f \text{ in } \Omega. \tag{1.7}$$

The homogenized coefficients $A^* = (a^*_{kl})$ *are given by*

$$a^*_{kl} = \frac{1}{|Y|}\int_Y a_{kl}(y)\,dy + \frac{1}{|Y|}\int_Y a_{kp}(y)\frac{\partial w^l}{\partial y_p}\,dy, \tag{1.8}$$

where the corrector $w^l \in H^1_\sharp(Y)/\mathbb{R},\ l = 1, 2, \ldots, d$ *satisfies the cell problem:*

$$-\operatorname{div}(A(e_l + \nabla w^l)) = 0 \text{ in } Y. \tag{1.9}$$

Various methods exist for obtaining homogenization limits such as the formal method of two-scale asymptotic expansions [SP80, BP89, BLP11], method of oscillating test functions [CK97], two-scale convergence [Ngu89, All92], gamma convergence [DM93], Bloch wave method [CV97], method of periodic unfolding [CDG18].

Further, for a broad introduction to the area of homogenization see [BP89, BLP11, JKO94, BD98, CD99, All02, MK06, CPS07, Tar09]. For an exhaustive account of quantitative theory of homogenization for periodic and stochastic media see [She18, AKM19] respectively.

1.2 Periodic homogenization by Bloch wave method

Differential equations with periodic coefficients have been studied by mathematicians such as Hill, Mathieu and Floquet since 19th century and the equations as well as the theory bear their name. F. Bloch rediscovered aspects of this theory for Schrödinger operator with a periodic potential in connection with solid state physics. The Bloch wave method, developed by Conca and Vanninathan [CV97], uses the spectral theory of periodic differential operators in $L^2(\mathbb{R}^d)$ to characterize the homogenization limit.

In the context of homogenization, some early works where Bloch waves make an appearance are [Sev81, Tur82, Zhi89, SS91] The book [CPV95] contains a wealth of information on Bloch waves as applied to fluid-structure interaction problems. The Bloch wave homogenization was formalized as a framework for periodic homogenization in [CV97]. Ever since, this framework has been applied to non-selfadjoint problems [SGV04], system of elasticity [SGV05], Maxwell's system [SEK+05], domain with holes [DOS09], Stokes system [ACFO07, AGV17], etc. It has also been combined successfully with the theory of two-scale convergence to study a variety of evolution problems [ACP+04, AP05, APR11, APR13]. More applications to dispersion limits appear in [DLS14, ABV16, BG19].

The works cited above mostly deal with qualitative aspects of homogenization. Quantitative aspects of Bloch wave homogenization were first studied through the Bloch approximation in [COV02, COV05]. Norm resolvent estimates were obtained in [BS04] using Bloch decomposition and perturbation theory. Operator theoretic techniques have further been exploited to study critical-contrast media in [CEK19].

The process of Bloch wave homogenization may be split into the following steps:

1. Spectral decomposition of the periodic operator using Bloch waves.
2. Convert the periodic equation into a family of elliptic equations in Bloch space.
3. Regularity of the Bloch eigenpairs in the Bloch parameter.
4. Localization and passage to limit.

1.2.1 Direct integral decomposition

The differential operators that arise in mathematical physics are usually self-adjoint with compact resolvents, however, a differential operator in $L^2(\mathbb{R}^d)$ rarely have compact resolvent. Hence, spectral theory of differential operators in $L^2(\mathbb{R}^d)$ is not very well developed. In a first step, under the additional structural assumption of periodicity, the operator $\mathcal{A}$ will be decomposed as a fibered operator called a "Hilbert bundle". This decomposition is called a direct integral decomposition. In a second step, the spectrum of fibered operators is given by Bloch eigenvalues and they can be diagonalized using Bloch eigenfunctions. Moreover, the spectrum of $\mathcal{A}$ is found to be the union of Bloch eigenvalues.

A rigorous development of direct integral decomposition may be found in [Mau68]. In this abstract theory, the symmetries associated with translation operators and the operator $\mathcal{A}$ induce a decomposition of the Hilbert space $L^2(\mathbb{R}^d)$

and a corresponding decomposition of the operator $\mathcal{A}$ as a direct integral whose fibers act distributively over the fibers of the decomposed Hilbert space.

Physically, the motivation comes from asking what kind of waves are supported by a periodically heterogeneous medium. It is known that plane waves are supported in a homogeneous medium. This physical statement may be interpreted to mean that plane waves act as generalized eigenfunctions for the Laplacian in $\mathbb{R}^d$. Therefore, we may ask what kind of eigenfunctions ψ exist for periodic operators. The answer to this question is motivated by the linear algebra fact that a family of commuting diagonalizable matrices can be diagonalized simultaneously. A periodic operator commutes with translation operators corresponding to translations by periods. These translation operators, given by $T_p\psi(y) := \psi(y+p)$ for $p \in \mathbb{Z}^d$, are unitary and therefore their eigenvalues are complex numbers of modulus 1. Hence, the eigenfunctions satisfy the following property:

$$T_p\psi(y) = \psi(y + 2\pi p) = e^{2\pi i\eta\cdot p}\psi(y), \;\; p \in \mathbb{Z}^d.$$

The functions satisfying the second equality are known as (η, Y)-periodic functions. This property is invariant under $\mathbb{Z}^d$-shifts of η, hence, without loss of generality we make the restriction that η should vary in $\eta \in \left[-\frac{1}{2}, \frac{1}{2}\right)^d$.

Clearly, there is a one-to-one correspondence between (η, Y)-periodic and Y-periodic functions. This is achieved by the following transformation

$$\psi(y) = e^{i\eta\cdot y}\phi(y),$$

where ψ is (η, Y)-periodic and ϕ is Y-periodic. The above relation is called the Floquet ansatz. In fact, every (η, Y)-periodic function can be written as a product of a periodic function and $e^{i\eta\cdot y}$. Hence, looking for (η, Y)-periodic eigenfunctions of $\mathcal{A}$ is the same as looking for Y-periodic eigenfunctions of the new 'shifted' operator

$$\mathcal{A}(\eta) = e^{-i\eta\cdot y}\mathcal{A}e^{i\eta\cdot y} = -\left(\frac{\partial}{\partial y_k} + i\eta_k\right)a_{kl}(y)\left(\frac{\partial}{\partial y_l} + i\eta_l\right), \tag{1.10}$$

This is an unbounded operator in $L^2_\sharp(Y)$, the space of all $L^2_{loc}(\mathbb{R}^d)$ functions that are Y-periodic.

The heuristics developed above can be formalized mathematically as follows. We shall refer to Y as the basic cell for the lattice $2\pi\mathbb{Z}^d$ and $Y' := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ as the basic cell for the dual lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$.

1. **Direct integral decomposition of $L^2(\mathbb{R}^d)$:** Given $g \in \mathcal{D}(\mathbb{R}^d)$, we define its Gelfand transform as
$$g_\sharp(y,\eta) = \sum_{p\in\mathbb{Z}^d} g(y+2\pi p)e^{-i(y+2\pi p)\cdot\eta}.$$
This is a function in the Bochner space $L^2(Y', L^2_\sharp(Y))$, which is the space of all $L^2_\sharp(Y)$-valued maps $\eta \in Y' \mapsto u_\sharp(\cdot,\eta)$ such that the norm function $\eta \mapsto \|u_\sharp(\eta)\|_{L^2_\sharp(Y)}$ is in $L^2(Y')$. The map $g \mapsto g_\sharp$ is an isometry from $\mathcal{D}(\mathbb{R}^d)$, equipped with the L^2-norm, to $L^2(Y', L^2_\sharp(Y))$ and hence it may be extended to a unitary isomorphism from $L^2(\mathbb{R}^d)$ to $L^2(Y', L^2_\sharp(Y))$. The space $L^2(Y', L^2_\sharp(Y))$ is the direct integral with the constant fiber $\int_{Y'}^{\oplus} L^2_\sharp(Y)\,d\eta$.

2. **Direct integral decomposition of $\mathcal{A}$:** Then, the operator $\mathcal{A}$ in $L^2(\mathbb{R}^d)$ is unitarily equivalent to the fibered operator
$$\int_{Y'}^{\oplus} \mathcal{A}(\eta)\,d\eta$$
in the Bochner space $L^2(Y', L^2_\sharp(Y))$. If $u \in D(\mathcal{A})$ and $u = \int_{Y'} u_\sharp(\cdot;\eta)\,d\eta$ then $u_\sharp \in D(\mathcal{A}(\eta))$ and the operator $\int_{Y'}^{\oplus} \mathcal{A}(\eta)\,d\eta$ acts on $u_\sharp \in L^2(Y', L^2_\sharp(Y))$ distributively, i.e., $\left(\int_{Y'}^{\oplus} \mathcal{A}(\eta)\,d\eta\right)u_\sharp(\cdot,\eta) = \int_{Y'}^{\oplus} \mathcal{A}(\eta)u_\sharp(\cdot,\eta)\,d\eta$.

3. The **spectrum of** $\mathcal{A}$ is union of the spectra of the shifted operator $\mathcal{A}(\eta)$ as η varies over Y'.

This situation is expressed in the commutative diagram below.

$$\begin{array}{ccc}
L^2(\mathbb{R}^d) & \xrightarrow{\mathcal{A}} & L^2(\mathbb{R}^d) \\
\big\downarrow \mathcal{G} & & \big\downarrow \mathcal{G} \\
L^2(Y', L^2_\sharp(Y)) & \xrightarrow{\int_{Y'}^{\oplus} \mathcal{A}(\eta)\,d\eta} & L^2(Y', L^2_\sharp(Y))
\end{array}$$

Here, $\mathcal{G}$ denotes the Gelfand transform. The proof of these facts may be found in [RS80, Fel00, Fol16].

1.2.2 Bloch Decomposition

In the previous subsection, the space $L^2(\mathbb{R}^d)$ was identified with $L^2(Y'; L^2_\sharp(Y))$ and the operator $\mathcal{A}$ was identified with a fibered operator. In this subsection, we shall solve the eigenvalue problem associated with the fibers of the direct integral. These eigenvalues will be called as Bloch eigenvalues.

The eigenvalue problem associated to the operator family $\mathcal{A}(\eta)$ is posed in the space

$$H^1_\sharp(Y) = \{\phi \in H^1_{loc}(\mathbb{R}^d) : \phi(y + 2\pi p) = \phi(y) \text{ a.e. } y \in \mathbb{R}^d,\ p \in \mathbb{Z}^d\}.$$

The following bilinear form is associated to $\mathcal{A}(\eta)$:

$$a(\eta)(u, v) = \int_Y a_{kl}(y)\left(\frac{\partial u}{\partial y_l} + i\eta_l\right)\overline{\left(\frac{\partial v}{\partial y_k} + i\eta_k\right)}\, dy.$$

In order to solve the eigenvalue problem for $\mathcal{A}(\eta)$, we prove that it has a compact resolvent. We first look at the problem of invertibility, i.e.,

Given $f \in L^2_\sharp(Y)$, find $u \in H^1_\sharp(Y)$ satisfying $a(\eta)(u, v) = (f, v)\ \forall\ v \in H^1_\sharp(Y)$.

Now, although the bilinear form $a(\eta)$ is not coercive, a translation of $a(\eta)$ in the lowest order is coercive. It will suffice to prove the following Gårding-type inequality.

Theorem 1.2. *There are positive numbers* α, β *such that for all* $u \in H^1_\sharp(Y)$*, the following equality holds*

$$a(\eta)(u, u) + \alpha\|u\|^2_{L^2_\sharp(Y)} \geq \beta\|u\|^2_{H^1_\sharp(Y)}. \tag{1.11}$$

A proof of a similar inequality in the context of elasticity equation may be found in [Ros87, BLP11, CPV95]. As a consequence, by the standard spectral theorem of compact self-adjoint operators, $\mathcal{A}(\eta)$ has an increasing sequence of eigenvalues and an orthonormal basis of $L^2_\sharp(Y)$ composed of corresponding eigenfunctions, for every $\eta \in Y'$. Let $(\lambda_m(\eta))_{m=1}^\infty$ denote the sequence of increasing eigenvalues for $\mathcal{A}(\eta)$, counting multiplicity. Let $(\phi_m(\eta))_{m=1}^\infty$ be the corresponding eigenfunctions. The functions $\eta \mapsto \lambda_m(\eta)$ are known as the Bloch eigenvalues of the operator $\mathcal{A}$. They are also often called as *band functions*. The corresponding eigenfunctions are called Bloch eigenfunctions. This allows us to further decompose the direct integral as a diagonalization. In particular, the Bloch transform maps $L^2(\mathbb{R}^d)$ to $L^2(Y', \ell_2(\mathbb{N}))$

Theorem 1.3. *Let* $g \in L^2(\mathbb{R}^d)$*. Define the* m^{th} *Bloch coefficient of* g *by*

$$\mathcal{B}_m g(\eta) := \int_{\mathbb{R}^d} g(y)e^{-iy\cdot\eta}\overline{\phi_m(y;\eta)}\, dy,\ m \in \mathbb{N},\ \eta \in Y'. \tag{1.12}$$

Then, the following inverse formula holds

$$g(y) = \int_{Y'} \sum_{m=1}^\infty \mathcal{B}_m g(\eta)\phi_m(y;\eta)e^{iy\cdot\eta}\, d\eta. \tag{1.13}$$

Further,

$$||g||^2_{L^2(\mathbb{R}^d)} = \sum_{m=1}^{\infty} \int_{Y'} |\mathcal{B}_m g(\eta)|^2 \, d\eta. \tag{1.14}$$

For $g \in D(\mathcal{A})$,

$$\mathcal{B}_m(\mathcal{A}g)(\eta) = \lambda_m(\eta)\mathcal{B}_m g(\eta). \tag{1.15}$$

The proof may be found in [SGV04] and [SGV05].

Now that the Bloch decomposition of the operator $\mathcal{A}$ is achieved, we shall repeat this process to bring about the Bloch decomposition at the $\epsilon - scale$, i.e., the Bloch decomposition of operator $\mathcal{A}^\epsilon = -\text{div}(A^\epsilon(x)\nabla)$. This operator is unitarily equivalent to a direct integral, given by

$$\int_{Y'/\epsilon}^{\oplus} \mathcal{A}^\epsilon(\xi)\, d\xi. \tag{1.16}$$

We shall denote the eigenvalues and eigenfunctions of $\mathcal{A}^\epsilon(\xi)$ by $(\lambda_m^\epsilon(\xi), \phi_m^\epsilon(x,\xi))_{m\in\mathbb{N}}$. The Bloch eigenvalues and Bloch eigenfunctions of $\mathcal{A}$ and $\mathcal{A}^\epsilon$ are related by the following equations.

$$\lambda_m^\epsilon(\xi) = \epsilon^{-2}\lambda_m(\epsilon\xi),\ \phi_m^\epsilon(x;\xi) = \phi_m(x/\epsilon;\epsilon\xi). \tag{1.17}$$

This yields a Bloch decomposition of $L^2(\mathbb{R}^d)$ at the ϵ-scale.

Theorem 1.4. *Let* $g \in L^2(\mathbb{R}^d)$. *Define the* m^{th} *Bloch coefficient of* g *as*

$$\mathcal{B}_m^\epsilon g(\xi) := \int_{\mathbb{R}^d} g(x) e^{-ix\cdot\xi}\overline{\phi_m^\epsilon(x;\xi)}\, dx,\ \ m \in \mathbb{N},\ \ \xi \in Y'/\epsilon. \tag{1.18}$$

Then, the following inverse formula holds

$$g(x) = \int_{Y'/\epsilon} \sum_{m=1}^{\infty} \mathcal{B}_m^\epsilon g(\xi)\phi_m^\epsilon(x;\xi) e^{ix\cdot\xi}\, d\xi. \tag{1.19}$$

Further,

$$||g||^2_{L^2(\mathbb{R}^d)} = \sum_{m=1}^{\infty} \int_{Y'/\epsilon} |\mathcal{B}_m^\epsilon g(\xi)|^2 \, d\xi. \tag{1.20}$$

For $g \in D(\mathcal{A}^\epsilon)$,

$$\mathcal{B}_m^\epsilon(\mathcal{A}^\epsilon g)(\xi) = \lambda_m^\epsilon(\xi)\mathcal{B}_m^\epsilon g(\xi). \tag{1.21}$$

Then, as a consequence of the representation (1.16), an equation like $\mathcal{A}^\epsilon u^\epsilon = f$, where $f \in L^2(\mathbb{R}^d)$, can be written as a cascade of equations in the Bloch space, viz.,

$$\begin{aligned} \lambda_1^\epsilon(\xi)\mathcal{B}_1^\epsilon(u^\epsilon)(\xi) &= \mathcal{B}_1^\epsilon(f) \\ \lambda_2^\epsilon(\xi)\mathcal{B}_2^\epsilon(u^\epsilon)(\xi) &= \mathcal{B}_2^\epsilon(f) \\ &\vdots \\ \lambda_m^\epsilon(\xi)\mathcal{B}_m^\epsilon(u^\epsilon)(\xi) &= \mathcal{B}_m^\epsilon(f) \\ &\vdots \end{aligned} \tag{1.22}$$

Now, the homogenization can be achieved by passing to the limit in these equations in the Bloch space. However, in order to achieve that, more properties of the Bloch eigenvalues and the Bloch coefficients are required.

1.2.3 Regularity of Bloch waves in the dual variable

By an application of the min-max principle, the Bloch eigenvalues can be proved to be Lipschitz continuous functions of the dual variable η. The proof may be found in [CV97].

Theorem 1.5. *[CV97] For* $m \geq 1$, $\eta \mapsto \lambda_m(\eta)$ *is a Lipschitz continuous function of* η.

In contrast, there is a good deal of choice present, when it comes to the eigenfunctions. Wilcox [Wil78] proved that the Bloch eigenfunctions of operators of the form $-\Delta + V$, where V is periodic, could be chosen measurably and that they could be chosen to be real analytic outside of a measure zero set. The Bloch decomposition theorem of the previous section requires that the Bloch eigenfunctions be measurable in the dual parameter.

These results, however, are insufficient for the proof of homogenization. The following result pulls us through, a proof of which may be found in [CV97].

Theorem 1.6. *[CV97] There is a small ball* $B_\delta(0)$ *such that*

- $\eta \mapsto \lambda_1(\eta)$ *is analytic for* $\eta \in B_\delta(0)$.
- *There is a choice of the corresponding eigenvector* $\eta \mapsto \phi_1(.,\eta)$ *satisfying*

$$\eta \in B_\delta(0) \mapsto \phi_1(\cdot,\eta) \in H^1_\sharp(Y) \text{ is analytic and}$$
$$\phi_1(y,0) = |Y|^{-1/2}, \text{ a constant independent of } y.$$

This theorem depends on the fact that the first eigenvalue of the operator $\mathcal{A}$ is simple, and therefore, remains simple in a neighbourhood of 0. It involves use of an implicit function theorem in the analytic category and in an infinite dimensional setup by way of an infinite dimensional determinant. The regularity of the eigenvectors is a more involved task and makes use of an analytic projection operator. These results may also be proved by using the Kato-Rellich theorem, which will be discussed in Section 1.3.

1.2.4 Identification of homogenized tensor

The homogenized tensor and the correctors can be expressed in terms of derivatives of Bloch eigenvalues and Bloch eigenfunctions. This is the content of the next theorem, whose proof may be found in [CV97] or [SGV04]. Such calculations are also performed in [BLP11].

Theorem 1.7. *The first Bloch eigenvalue $\lambda_1(\eta)$ and eigenfunction $\phi_1(\eta)$ of $\mathcal{A}$ satisfy:*

1. $\lambda_1(0) = 0$.
2. *The eigenvalue $\lambda_1(\eta)$ has a critical point at $\eta = 0$, i.e.,*
$$\frac{\partial \lambda_1}{\partial \eta_s}(0) = 0, \forall s = 1, 2, \dots, d. \tag{1.23}$$
3. *For $s = 1, 2, \dots, d$, the derivative of the eigenvector $(\partial \phi_1/\partial \eta_s)(0)$ satisfies: $(\partial \phi_1/\partial \eta_s)(y; 0) - i\phi_1(y; 0)w^s(y)$ is a constant in y, where w^s is defined in (1.9).*
4. *The Hessian of the first Bloch eigenvalue at $\eta = 0$ is twice the homogenized matrix a^*_{kl}:*
$$\frac{1}{2}\frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_l}(0) = a^*_{kl}. \tag{1.24}$$

Remark 1.8.

1. It can be seen easily that $\eta \mapsto \lambda_m(\eta)$ are periodic functions with Y' as the basic periodicity cell.
2. The first Bloch eigenvalue is an even function with respect to the origin, i.e., $\lambda_1(\eta) = \lambda_1(-\eta)$ for all $\eta \in Y'$. Equation (1.23) can be seen as a consequence of this fact.

1.2.5 Localization and passage to limit

For the purpose of homogenization, we need one more result which describes the weak limit of the first Bloch coefficient of a sequence of weakly convergent functions.

Theorem 1.9. *Let* g^ϵ *be a sequence of functions in* $L^2(\mathbb{R}^d)$ *with uniform (in* ϵ*) compact support in* $K \subseteq \mathbb{R}^d$ *such that* $g^\epsilon \rightharpoonup g$ *in* $L^2(\mathbb{R}^d)$*-weak for some function* $g \in L^2(\mathbb{R}^d)$*. Then it holds that*

$$\chi_{\epsilon^{-1}Y}\mathcal{B}_1^\epsilon g^\epsilon \rightharpoonup \widehat{g}$$

in $L^2_{loc}(\mathbb{R}^d_\xi)$*-weak, where* $\widehat{g}$ *denotes the Fourier transform of* g*.*

Now, we note that the homogenized equation will be recovered by passing to the limit in the first equation in the cascade (1.22) and that the rest of the equations will not contribute to the limit equation. However, the cascade is written for an equation posed in $\mathbb{R}^d$. In order to obtain homogenized equation for an equation posed for a domain $\Omega \subseteq \mathbb{R}^d$, we first need to localize it as follows. Let ψ_0 be a fixed element in $\mathcal{D}(\Omega)$. Since u^ϵ satisfies $\mathcal{A}^\epsilon u^\epsilon = f$ in Ω, $\psi_0 u^\epsilon$ satisfies

$$\mathcal{A}^\epsilon(\psi_0 u^\epsilon)(x) = \psi_0 f(x) + h_1^\epsilon(x) + h_2^\epsilon(x) \text{ in } \mathbb{R}^d, \tag{1.25}$$

where

$$\begin{aligned} h_1^\epsilon(x) &:= -\frac{\partial\psi_0}{\partial x_k}(x) a_{kl}^\epsilon(x)\frac{\partial u^\epsilon}{\partial x_l}(x), \\ h_2^\epsilon(x) &:= -\frac{\partial}{\partial x_k}\left(\frac{\partial\psi_0}{\partial x_l}(x) a_{kl}^\epsilon(x) u^\epsilon(x)\right). \end{aligned}$$

On applying the first Bloch coefficient to the equation (1.25), we obtain the equation:

$$\lambda_1^\epsilon(\xi)\mathcal{B}_1^\epsilon(\psi_0 u^\epsilon)(\xi) = \mathcal{B}_1^\epsilon(\psi_0 f)(\xi) + \mathcal{B}_1^\epsilon h_1^\epsilon(\xi) + \mathcal{B}_1^\epsilon h_2^\epsilon(\xi) \text{ in } \mathbb{R}^d. \tag{1.26}$$

In order to pass to the limit in equation (1.26), we shall expand λ_1^ϵ as a power series in a neighbourhood of $\xi = 0$. Also, the limits of $\mathcal{B}_1^\epsilon h_1^\epsilon$ and $\mathcal{B}_1^\epsilon h_2^\epsilon$ are identified. Finally, a limit equation is found in Bloch space. The homogenized equation in physical space is found by taking the inverse Fourier transform. All of the details may be found in [CV97].

1.2.6 Error estimates

The passage to limit and recovery of homogenized tensor, as described above, is part of the qualitative theory of homogenization. Another important aspect of homogenization is the quantitative theory, i.e., obtaining rates of convergence for the L^2-convergence of u^ϵ to u^*, as defined in Theorem 1.1. By making use of the correctors, the L^2-convergence may be updated to H^1-convergence by using a better approximation, $u^\epsilon \approx u^* + \epsilon w^s \frac{\partial u^*}{\partial x_s}$.

In the early literature, rates have been obtained for scalar equations by making use of the maximum principle [BLP11] and the boundedness of correctors due to De Giorgi-Nash-Moser regularity. Those requirements clearly fail for systems. As a consequence, it is standard practice to take Hölder continuous coefficients in the case of systems, so that Schauder theory is applicable. One of the advantages of Bloch wave homogenization is to obtain such error estimates under optimal hypotheses on regularity of coefficients.

In the theory of Bloch wave homogenization, error estimates were first obtained in [COV02, COV05] by proposing an approximation different from the one mentioned above. The authors call this approximation the Bloch approximation which is defined by

$$\theta^\epsilon(x) = \int_{Y'/\epsilon} \widehat{u^*}(\xi) e^{ix\cdot\xi} \phi_1^\epsilon(x;\xi)\, d\xi$$

and it is proved that $|u^\epsilon - \theta^\epsilon|_{H^1} = O(\epsilon)$.

Error estimates for homogenization have also been obtained by Birman and Suslina [BS04] in the form of order-sharp resolvent estimates by the spectral approach, i.e., they use the Bloch decomposition and perturbation theory to obtain rates of convergences in the full space.

$$||(\mathcal{A}^\epsilon + I)^{-1} - (\mathcal{A}^* + I)^{-1}||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq C\epsilon.$$

These are also referred to as norm resolvent estimates. Their methods are not restricted to scalar equations. Further, recently Suslina and co-authors have obtained operator error estimates in bounded domains as well [PS12, Sus13, Sus17]. These developments have been foreshadowed by [Sev81, Zhi89].

1.2.7 Comments on the Bloch wave method

Bloch wave homogenization has several features that make it ideal to study a wide variety of physical phenomena. These features include

1. Bloch wave homogenization has allowed the interpretation of homogenization as a *spectral threshold phenomena* [BS04]. This has been very fruitful as it has led to the notion of internal edge homogenization [BS06].

2. In the case of a multiple spectral edge, such as elasticity operator [SGV05], Stokes system [AGV17], the homogenized tensor appears through a relation called the *propagation condition* involving directional derivatives of the Bloch eigenvalues. This may be interpreted as follows: the directional Hessian gives the speed of propagation of waves in the homogenized medium in corresponding directions. Such an interpretation is not forthcoming in other methods of homogenization for system of equations.

3. In long time asymptotics for the heterogeneous wave equation, the fourth order derivative of the first Bloch eigenvalue at 0, called the dispersion tensor, plays an important role [COV06]. This leads to a fourth order equation for the homogenized medium [DLS14, ABV16]. The dispersion tensor differs from the fourth order tensor that appears by way of the two-scale asymptotic expansions [ABV16].

4. The homogenized tensor appears in a more natural manner in this framework as the second order coefficient in the power series expansion of the first Bloch eigenvalue at $\xi = 0$. This is even more striking in the setting of critical sized perforated domains where the so-called "strange term" [DOS09] appears through the first order term in the power series expansion of the first Bloch eigenvalue at $\xi = 0$.

5. The passage to limit in homogenization is greatly simplified in this framework. In particular, the homogenization limit is obtained as a weak limit of the Bloch transformed equation, which is an algebraic equation instead of a differential equation [CV97].

6. It opens up avenues of research into the interaction of homogenization theory with other physical phenomena such as creation of spectral gaps [Zhi04], non-locality [MCFK, CC16], resonance [CEK19], Anderson localization [Ves02], etc.

7. The operator error estimates in [Sus13] do not require Hölder continuity of the coefficients. This is relevant because microstructures that appear in applications are not expected to have any regularity. In fact, the simple configuration of two-phase media is only measurable and bounded.

1.3 Parametrized eigenvalue problems

In this section, we treat Bloch wave homogenization as a gateway to eigenvalue problems involving multiple parameters. Eigenvalue problems, that depend on multiple parameters, arise in a variety of mathematical models, for example, Bloch waves (Solid State Physics), Hyperbolic systems of PDEs (Wave motion), etc. Parametrized eigenvalue problems have been studied by Rellich and Kato under the heading of Perturbation Theory. Sometimes, it is also studied as Bifurcation Theory. We saw in Theorem 1.6 that the first Bloch eigenvalue is analytic in a neighbourhood of $\eta = 0$. Such a theorem is proved by appealing to perturbation theory. In this section, we shall discuss some finer points related to perturbation theory.

Generally speaking, the regularity of the family of operators is not transferred to the eigenvalues and eigenvectors. When a family of self-adjoint operators depends analytically on one real or complex variable, Rellich [Rel69] has proved that there exists an arrangement of the eigenvalues which is analytic. This arrangement does not correspond to the usual increasing order of eigenvalues unless the eigenvalue, in question, is simple. The analytic eigenvectors, so obtained, also only correspond to the specific arrangement of eigenvalues obtained from the theorem. When we go to families of operators depending on multiple parameters, we may not have analyticity unless the eigenvalue(s) are simple, i.e., analyticity may be lost at the points of eigenvalue crossings.

Let us illustrate these phenomena using some examples. Consider the simplest one-dimensional family of matrices, viz.,

$$\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \; x \in \mathbb{R}.$$

The ordered eigenvalues are known to be $|x|$ and $-|x|$. These, however, are not even differentiable. Rellich comes to the rescue, and the rearranged eigenvalues x and $-x$ are analytic. The corresponding eigenvectors $(1\ 0)^T$ and $(0\ 1)^T$ are constant, therefore, analytic. We cannot choose the eigenvectors to be a certain vector at $x = 0$. Clearly, at $x = 0$, any vector would be an eigenvector of the given matrix. But, a prescribed eigenvector at 0 cannot be continued for other values of x. As Rellich points out, "the perturbation method itself selects them."

As an example with multiple parameters, consider

$$\begin{pmatrix} x & y \\ y & -x \end{pmatrix}, \; x, y \in \mathbb{R}.$$

The ordered eigenvalues are known to be $\sqrt{x^2+y^2}$ and $-\sqrt{x^2+y^2}$, and no arrangement of these would make them differentiable around 0.

In the next subsection, we review how simplicity assures us of regularity of the eigenpairs.

1.3.1 Simplicity implies Analyticity

One situation where the number of parameters is immaterial for obtaining regularity of eigenvalues is when the eigenvalues are simple. The main tool for the proof is the implicit function theorem in various categories. Implicit function theorem is a local result and any proof following from that premise will only be local in nature. We shall begin by proving such a result in finite dimensions.

Theorem 1.10. *Let* $A(t)$ *be a continuously differentiable square matrix valued function of real variable* t. *Suppose that* $A(0)$ *has an eigenvalue* λ_0 *of multiplicity one. Then, for* t *small enough,* $A(t)$ *has an eigenvalue* $\lambda(t)$ *that depends differentiably on* t, *such that* $\lambda(0) = \lambda_0$. *Further, an eigenvector* $x(t)$ *corresponding to* $\lambda(t)$ *can be chosen to depend on* t *differentiably.*

Proof. The characteristic polynomial of $A(t)$ is $\det(sI - A(t)) = p(s,t)$ is a polynomial of degree n with C^1 coefficients. The eigenvalue λ_0 at 0 is simple. Therefore, $p(\lambda_0, 0) = 0$ and $\frac{\partial p}{\partial s}(\lambda_0, 0) \neq 0$. By application of the Implicit function theorem, there is a C^1 function $\lambda(t)$ defined in a neighbourhood of 0, that satisfies $p(\lambda(t), t) = 0$. Eigenvector may be constructed locally, using appropriate minors of $A(t)$. □

Moving on to unbounded self-adjoint operators with compact resolvent on infinite dimensional Hilbert spaces, there are conditions under which we can define an infinite dimensional version of the determinant. This is an analytic function with zeros corresponding to the eigenvalues and the multiplicity preserved. This should be reminiscent of the Weierstrass factorization theorem which involves constructing an analytic function with a prescribed set of zeros. The theory holds for Hilbert-Schmidt operators and even for the class $\mathcal{I}_n$, which comprises of compact operators whose singular values are in the sequence space ℓ^n. Such a construction can be found in [Sik61b], [Sik61a], [Smi41], [RS78], [GK69], [DS88], among others. The rest of the proof follows in like fashion. The construction of eigenvectors could also be done using the infinite dimensional version of determinant and its minors.

However, one could also make use of the projection operator onto the subspace spanned by the first eigenvalue in a neighbourhood of 0. Such a projection operator can be defined using a version of residue calculus. Specifically, for a closed operator $\mathcal{A}$, if λ is an isolated point of the spectrum $\sigma(\mathcal{A})$, the integral

$$P_\lambda = -\frac{1}{2\pi i}\oint_{|\lambda-\mu|=r} (\mathcal{A}-\mu)^{-1} d\mu$$

exists, where r is the radius of a circle containing λ alone [RS78, p. 11]. For an analytic family of operators $\mathcal{A}(\eta)$, the family of projection operators, $P_\lambda(\eta)$, is also analytic and if the eigenspace is one-dimensional, we obtain an analytic choice of eigenvectors by defining $\phi(y,\eta) = P_\lambda(\eta)\phi(y,0)$. This can be further normalized according to needs.

1.3.2 Single Parameter implies Analyticity

Simplicity is not the only criteria for regularity of eigenfunctions. If the family of operators depends on a single parameter, Rellich [Rel69] proves the following.

Theorem 1.11 (Rellich). *The eigenvalues of a Hermitian matrix $a_{kl}(x)$, whose coefficients are power series for small $|x|$, can be rearranged to be power series for small $|x|$.*

This result uses the concept of a Puiseux Series and the Hermitian hypothesis cannot be removed. Further, analytic eigenvectors can be constructed for the corresponding eigenvalues. As mentioned before, we cannot decide an initial value for the eigenvectors (at 0).

This result is extended for operators over infinite dimensional Banach spaces in Kato's treatise [Kat95]. Kato begins by defining two types of operator families.

Definition 1.12 (Kato). Let R be a connected open set in the complex plane and let $T(z)$, a closed operator with nonempty resolvent set, be given for each $z \in R$. We say that $T(z)$ is an **analytic family of type** A if

1. The operator domain $A(z)$ is some set D independent of z.

2. For each $\psi \in D$, $A(z)\psi$ is a vector valued analytic function of z.

Definition 1.13. Suppose that $a(z)$ is a family of bounded sesquilinear forms with domain H for each $z \in D_0$, where D_0 is a connected open set in $\mathbb{C}$, and that $a(z)[u]$ is holomorphic in D_0 for each fixed $u \in H$. Such a family $\{a(z)\}$ is called **bounded-holomorphic**.

The family of operators $A(z) \in \mathcal{B}(H)$ defined by $(A(z)u, v) = a(z)[u, v]$ is a bounded-holomorphic family of operators.

Definition 1.14. The numerical range of a form $\mathfrak{a}$ is defined as $\Theta(\mathfrak{a}) = \{\mathfrak{a}[z] : z \in D(\mathfrak{a}), \|z\| = 1\}$.

It is defined analogously for operators.

Definition 1.15.

1. The form $\mathfrak{a}$ on H is called **sectorial** if there are numbers $c \in \mathbb{R}$ and $\theta \in [0, \pi/2)$ such that
$$\Theta(\mathfrak{a}) \subset S_{c,\theta} := \{\lambda \in \mathbb{C} : |\arg(\lambda - c)| \leq \theta)\}.$$

2. The operator A is called sectorial, if $\Theta(A) \subset S_{c,\theta}$ for some $c \in \mathbb{R}$ and $\theta \in [0, \pi/2)$.

3. The operator A is said to be **m-sectorial** if $\Theta(A) \subset S_{c,\theta}$ and if A is closed and $R(A - \alpha I)$ is dense in H, where $\alpha \in \mathbb{C} \setminus S_{c,\theta}$. We call c a vertex and θ a corresponding semi-angle.

Definition 1.16 (Kato). $\mathfrak{a}(z)$ is called a **holomorphic family of type (a)** if

1. each $\mathfrak{a}(z)$ is sectorial and closed with domain D independent of z and dense in H,

2. $\mathfrak{a}(z)[u]$ is holomorphic for $z \in D_0$ for each $u \in D$.

Kato [Kat95, p.395] proves that a holomorphic family of type $(\mathfrak{a})$ generates a family of $m-$sectorial operators. These are called as a **holomorphic family of type** B.

Finally, for the two types of operator families, he proves the following regularity theorem, mirroring Rellich's result for matrices.

Theorem 1.17 (Kato). *Let $A(x)$ be a self adjoint holomorphic family of type A or B defined for x in a neighborhood of an interval $I_0 \subset \mathbb{R}$. Furthermore, let $A(x)$ have compact resolvent. Then all eigenvalues of $A(x)$ can be represented by functions which are holomorphic on I_0. More precisely, there is a sequence of scalar-valued functions $\mu_n(x)$ and a sequence of vector-valued functions $\phi_n(x)$, all holomorphic on I_0, such that for $x \in I_0$, the $\mu_n(x)$ represent all the eigenvalues of $A(x)$ counting multiplicities and the $\phi_n(x)$ are the corresponding eigenfunctions which form a complete orthonormal family in the underlying Hilbert space.*

For results of a similar kind and with different regularity, for example, Hölder continuity and for non-selfadjoint operators, see papers from Rainer and coauthors [Rai11, Rai13, Rai14, KM03, KMR12, AKML98, LR07].

1.3.3 Genericity of Simple Eigenvalues

Multiple parameters are unavoidable in most applications of interest such as propagation of singularities for hyperbolic systems of equations with multiple characteristics leading to novel phenomena such as conical refraction [Lax82], [Den88], stability of hyperbolic initial-boundary-value problems [MZ05] and Bloch waves for elasticity system [BS04], [SGV05]. In such situations, Rellich and Kato's result may not always be applicable (although a successful application for elasticity system is brought about in [SGV05]). Hence, an assumption of simplicity is useful in applications [ACP+04], [APR11], [APR13]. Even though the original problem might not involve simple eigenvalues, it is often true that a small perturbation of the original problem has simple eigenvalues. A property depending on a parameter in a topological space X is said to be *generic* in X if the set of parameters on which it does not hold is of first category in X. In particular, a generic property holds densely in X. In the literature, it has been shown that under perturbations of some relevant parameters like domain shape, coefficients, potentials etc, a multiple eigenvalue can be made simple. Below we review some results of this kind.

Albert [Alb75] proves that the eigenvalues of a linear self-adjoint elliptic differential operator on a compact smooth manifold are generically simple under perturbation of lowest order term.

Theorem 1.18 (Albert). *Let M be a compact, connected C^∞ manifold and L a linear, self-adjoint, elliptic differential operator on M with C^∞ coefficients. Let A_n be the set of all $p \in C^\infty(M)$ such that the first n eigenvalues of $L + p$ are simple. Then for every $m \in \mathbb{N}$, A_m is open in $C^\infty(M)$ and each A_{m+1} is dense in A_m. Therefore, the set of all $p \in C^\infty(M)$ such that the eigenvalues of $L + p$ are simple, is generic in $C^\infty(M)$.*

Making use of infinite dimensional versions of Sard's Theorem, Uhlenbeck [Uhl76] proves similar results.

Theorem 1.19 (Uhlenbeck). *Let M_n be a compact n-manifold and let L_b be a family of self-adjoint elliptic operators on M_n with the parameter $b \in U$ an open subset of a Banach space B. Suppose that the coefficients of L_b are C^k for large enough k, then the following properties are generic in B*

1. *L_b has one-dimensional eigenspaces.*
2. *zero is not a critical value of the eigenfunctions, restricted to the interior of the domain of the operator;*

3. *the eigenfunctions are Morse functions on the interior of M.*

Generic simplicity of the spectrum with respect to domain has been established and applied in proving stabilizability and controllability results for the plate equation [OZ00] and the Stokes system in two dimensions [OZ01] by Ortega and Zuazua.

Therefore, we expect that a multiple Bloch eigenvalue can be made simple by a perturbation in coefficients of the operator $\mathcal{A}$.

1.4 Regularity of spectral edges

The previous subsections deal with the analytic structure of the Bloch spectrum of the periodic operator $\mathcal{A}$. In this subsection, certain geometrical and topological aspects of the Bloch spectrum will be revealed through the notion of spectral edges, spectral gaps and eigenvalue crossings.

Recall that the m^{th} Bloch eigenvalue of the operator $\mathcal{A}$ is denoted by $\lambda_m(\eta)$. Let $\sigma_m^- = \min_{\eta\in Y'} \lambda_m(\eta)$ and $\sigma_m^+ = \max_{\eta\in Y'} \lambda_m(\eta)$, then, the spectrum of the operator $\mathcal{A}$ is given by $\bigcup_{m\in\mathbb{N}}[\sigma_m^-, \sigma_m^+]$. Therefore, it is a union of closed intervals, which may overlap. However, it may also be written as $[0,\infty) \setminus \sqcup_{j=1}^N (\mu_j^-, \mu_j^+)$, where N takes values in $\mathbb{N} \cup \{\infty\}$. The pairwise disjoint intervals (μ_j^-, μ_j^+) are known as **spectral gaps** and $(\mu_j^\pm)_{j=1}^N$ are known as **spectral edges**. The schematic diagram in Fig. 1.3 shows that not every $\sigma_m^\pm$ may be a spectral edge, even though the corresponding Bloch eigenvalue is simple.

The Bloch eigenvalues are functions of the dual parameter η. The dependence of the Bloch eigenvalues and eigenfunctions on the parameter η has been addressed in Section 1.3 where we noted that the study of parametrized eigenvalue problems is an active area of research, even in finite dimensions [AKML98], [Rai14]. However, we will see that such questions regarding regularity properties of the Bloch eigenvalues in the parameter are also important in applications. In particular, the behaviour of the Bloch eigenvalues near a spectral edge determines a variety of physical phenomena; such as, in the theory of effective mass [AP05], Bloch wave method in homogenization [CV97] and Anderson localization [Ves02].

The importance of these regularity properties has led to the following conjectures about a spectral edge, that are expected to hold either for all relevant physical parameters or in a generic sense:

(R1) The spectral edge must be **simple**, i.e., it is attained by a single Bloch eigenvalue.

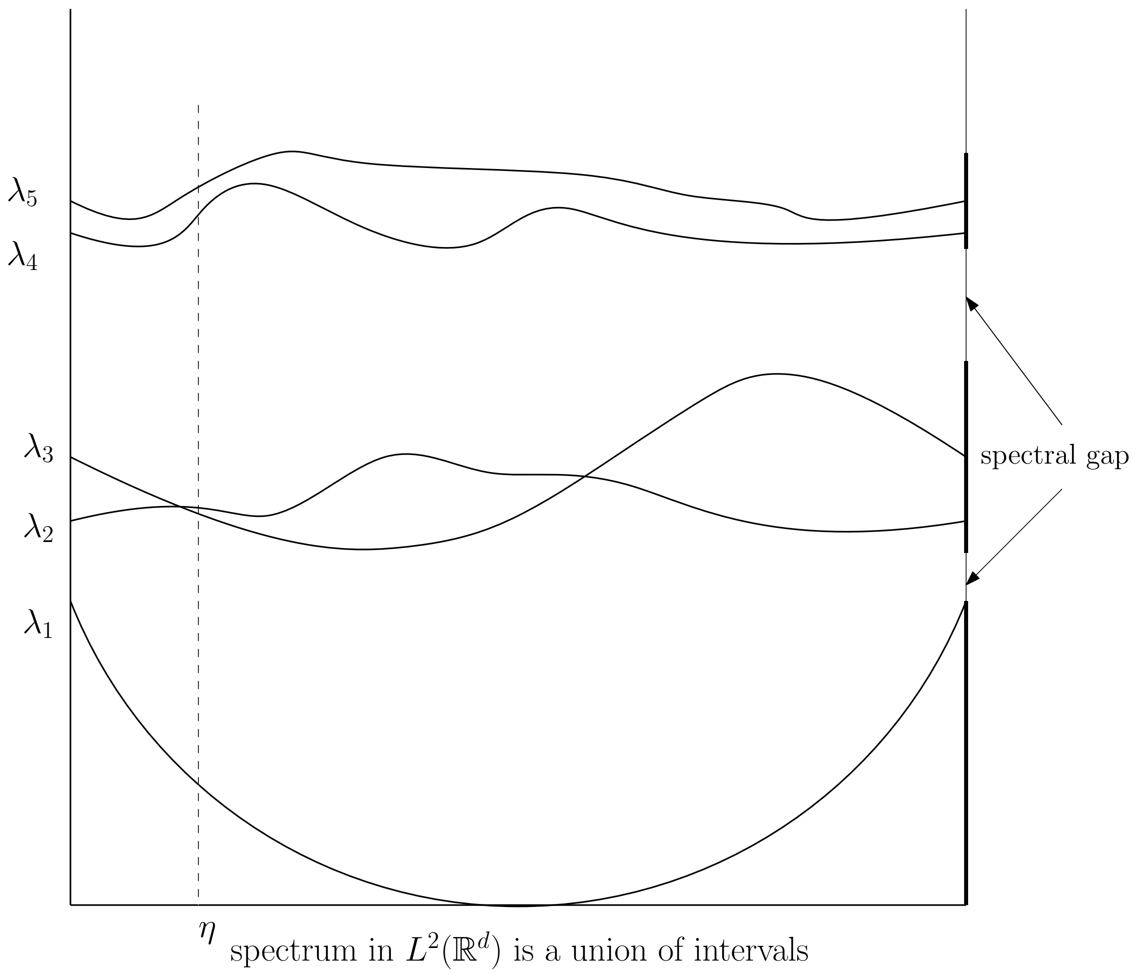


Figure 1.3: Bloch eigenvalues λ_4 and λ_5 are simple, but have no spectral gap between them.

(R2) The spectral edge must be **isolated**, i.e., it is attained at finitely many points in Y' by a Bloch eigenvalue.

(R3) The spectral edge must be **non-degenerate**, i.e., for some $m, r \in \mathbb{N}$, if the Bloch eigenvalue $\lambda_m(\eta)$ attains the spectral edge λ_0 at the points $\{\eta_j\}_{j=1}^r$, then the Bloch eigenvalue must satisfy, for $j = 1, 2, \ldots, r$,

$$\lambda_m(\eta) - \lambda_0 = (\eta - \eta_j)^T B_j (\eta - \eta_j) + O(|\eta - \eta_j|^3), \text{ for } \eta \text{ near } \eta_j,$$

where B_j are positive definite matrices.

While these features are readily available for the lowest Bloch eigenvalue corresponding to the divergence-type scalar elliptic operator, these properties may not be available for other spectral gaps of the same operator [Kuc16]. However, the following results are available regarding these properties:

- Klopp and Ralston [KR00] proved the simplicity of a spectral edge of Schrödinger operator $-\Delta + V$ under perturbation of the potential term.
- In two dimensions, spectral edges of a wide class of periodic elliptic operators are known to be isolated [FK18].
- Also, in two dimensions, a degenerate spectral edge of a Schrödinger operator can be made non-degenerate through a perturbation with a potential having a larger period [PS17].

The validity of hypotheses (R1), (R2), (R3) is usually assumed in the literature [Kuc16]; for example, in establishing Green's function asymptotics [KR12], [KKR17], for internal edge homogenization [BS06] and to establish localization for random Schrödinger operators [Ves02]. Local simplicity of Bloch eigenvalues is assumed in the study of diffractive geometric optics [APR11], [APR13] and homogenization of periodic systems [ACP$^+$04].

1.5 Internal edge homogenization

A major contribution of Birman and Suslina [BS04] to the theory of homogenization is its interpretation as a *spectral threshold effect*. For the operator $\mathcal{A}$, it is known that $\inf \sigma(\mathcal{A}) = 0$. This corresponds to the bottom edge of its spectrum. A non-zero spectral edge is called an internal edge. The notion of homogenization has been extended to internal edges in [Bir04], [BS06]. Correctors for internal edge homogenization are further developed in [SK09], [SK11].

In this subsection, we review the internal edge homogenization theorem of Birman and Suslina [BS06]. Consider the equation

$$-\nabla \cdot \left(A\left(\frac{x}{\epsilon}\right)\nabla u^\epsilon\right) + \vartheta^2 u^\epsilon = f \quad \text{in } \mathbb{R}^d, \tag{1.27}$$

corresponding to the operator $\mathcal{A}^\epsilon$. Let λ_0 denote an internal edge, corresponding to the upper endpoint of a spectral gap of $\mathcal{A}$ and let m be the smallest index such that the Bloch eigenvalue λ_m attains λ_0, then

$$\lambda_0 = \min_{\eta \in Y'} \lambda_m(\eta).$$

Birman and Suslina [BS06] make the following regularity assumptions on λ_0. These are exactly the properties of a spectral edge that are required in order to define effective mass in the theory of motion of electrons in solids [FK18].

(B1) λ_0 is attained by the m^{th} Bloch eigenvalue $\lambda_m(\eta)$ at finitely many points $\eta_1, \eta_2, \dots, \eta_N$.

(B2) For $j = 1, 2, \dots, N$, $\lambda_m(\eta)$ is simple in a neighborhood of η_j, therefore, $\lambda_m(\eta)$ is analytic in η near η_j.

(B3) For $j = 1, 2, \dots, N$, $\lambda_m(\eta)$ is non-degenerate at η_j, i.e.,

$$\lambda_m(\eta) - \lambda_0 = (\eta - \eta_j)^T B_j (\eta - \eta_j) + O(|\eta - \eta_j|^3), \text{ for } \eta \text{ near } \eta_j,$$

where B_j are positive definite matrices.

Under these assumptions, the internal edge homogenization theorem is proved.

Theorem 1.20 [BS06]. *Let $\mathcal{A}$ be the operator in $L^2(\mathbb{R}^d)$ defined by (1.1) and let λ_0 be an internal edge of the spectrum of $\mathcal{A}$. Let $\vartheta^2 > 0$ be small enough so that $\lambda_0 - \vartheta^2$ is in the spectral gap of $\mathcal{A}$. Assume conditions (B1), (B2), (B3). Let $\mathcal{A}^\epsilon$ denote the unbounded operator $-\nabla \cdot \left(A(\frac{x}{\epsilon})\nabla\right)$ defined in $L^2(\mathbb{R}^d)$. For $1 \le j \le N$, let $\psi_j(y, \eta_j) := \exp(iy \cdot \eta_j)\phi_j(y)$, where ϕ_j is the eigenvector corresponding to the eigenvalue $\lambda_0 = \lambda_m(\eta_j)$ of the operator $\mathcal{A}(\eta_j) = -(\nabla + i\eta_j) \cdot A(\nabla + i\eta_j)$. Then,*

$$||R(\epsilon) - R^0(\epsilon)||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(\epsilon) \quad \textit{as} \quad \epsilon \to 0, \quad \textit{where}$$

$$R(\epsilon) = \left(\mathcal{A}^\epsilon - (\epsilon^{-2}\lambda_0 - \vartheta^2)I\right)^{-1} \textit{ and}$$

$$R^0(\epsilon) := |Y| \sum_{j=1}^{N} [\psi_j^\epsilon] \left(-\nabla \cdot B_j \nabla + \vartheta^2 I\right)^{-1} [\overline{\psi_j^\epsilon}]$$

are bounded operators on $L^2(\mathbb{R}^d)$ and $\| \cdot \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}$ denotes the operator norm. Here, $[f]$ denotes the operation of multiplication by the function f.

1.6 Almost periodic homogenization

For $K = \mathbb{R}$ or $\mathbb{C}$, let $\mathrm{Trig}(\mathbb{R}^d; K)$ denote the space of all K-valued trigonometric polynomials of the form $P(y) = \sum_{j=1}^{N} a_j e^{iy\cdot\eta_j}$, defined for $y \in \mathbb{R}^d$.

Definition 1.21. A bounded continuous function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be uniformly almost periodic if it is the uniform limit of a sequence of real trigonometric polynomials, i.e., there exists a sequence $P_n(y) \in \mathrm{Trig}(\mathbb{R}^d; \mathbb{R})$ such that $\|u - P_n\|_\infty \to 0$ as $n \to \infty$.

Uniformly almost periodic functions are also known as Bohr almost periodic functions. The set of all Bohr almost periodic functions when equipped with the uniform norm is a Banach space denoted by $AP(\mathbb{R}^d)$.

Definition 1.22. The mean value of a function $u : \mathbb{R}^d \to \mathbb{R}$ in $L^1_{loc}(\mathbb{R}^d)$ is the following limit

$$\mathcal{M}(u) = \limsup_{L\to\infty} \frac{1}{|Y_L|} \int_{Y_L} u(y)\ dy, \tag{1.28}$$

where $Y_L = [-\pi L, \pi L)^d$ and $|\cdot|$ denotes its Lebesgue measure.

In fact, the limsup above is a limit for Bohr almost periodic functions [Bes55]. One can obtain the following class of functions by employing the notion of mean value. This class is larger than Bohr almost periodic functions.

Definition 1.23. A function $u \in L^2_{loc}(\mathbb{R}^d; \mathbb{C})$ is said to be Besicovitch almost periodic if there exists a sequence $P_n \in \mathrm{Trig}(\mathbb{R}^d; \mathbb{C})$ such that $\mathcal{M}(|u - P_n|^2) \to 0$ as $n \to \infty$.

On the set of all Besicovitch almost periodic functions, the quantity $(\mathcal{M}(|\cdot|^2))^{1/2}$ is a semi-norm. Given Besicovitch almost periodic functions f and g, we shall identify them if $\mathcal{M}(|f - g|^2) = 0$ to obtain a Hilbert space which will be denoted by $B^2(\mathbb{R}^d)$ with the inner product given by $\mathcal{M}(f \cdot \overline{g})$. The superscript 2 serves to remind us that one could very well define a Besicovitch analogue of L^p spaces.

Let us recall some interesting properties of almost periodic functions. In direct analogy with periodic functions, one can define a formal Fourier series for almost periodic functions [Bes55]. The trigonometric factors $e^{iy\cdot\eta}$ that appear in the Fourier series of an almost periodic function u correspond to all $\eta \in \mathbb{R}^d$ for which $\mathcal{M}(ue^{-iy\cdot\eta})$ is non-zero. Note that for a given function u, the set of all such η is countable. This set is called the set of frequencies of u and the $\mathbb{Z}$-module generated by these frequencies is denoted as $\mathrm{Mod}(u)$. Periodic functions are also almost periodic. In particular, continuous periodic functions belong to $AP(\mathbb{R}^d)$ whereas measurable and bounded periodic functions belong to $B^p(\mathbb{R}^d)$ for all p such that $1 \leq p < \infty$. Further, the mean value as defined in (1.28) coincides with the average or mean value of periodic functions over any basic cell of periodicity. Almost periodic functions u with a finitely generated $Mod(u)$ are called quasiperiodic functions. It is interesting to note that $AP(\mathbb{R}^d)$ and $B^2(\mathbb{R}^d)$ are examples of non-separable Banach spaces. More information about these function spaces may be found in [Bes55, LZ82, Cor09]. Further, a short but illuminating crash course on almost periodic functions may be found in [Shu78].

1.6.1 Almost Periodic Differential Operators

Consider the almost periodic second-order elliptic operator in divergence form given by

$$\mathcal{A}u := -\text{div}(A\nabla u) = -\frac{\partial}{\partial y_k}\left(a_{kl}(y)\frac{\partial u}{\partial y_l}\right), \tag{1.29}$$

where the coefficients satisfy the assumptions (A1), (A2), (A3). Further, in this section, the coefficients a_{kl} are also assumed to be uniformly almost periodic, i.e., $a_{kl} \in AP(\mathbb{R}^d)$.

Let Ω be an open set in $\mathbb{R}^d$. We are interested in the homogenization of the following equation posed in $H^1(\Omega)$

$$\mathcal{A}^\epsilon u^\epsilon := -\frac{\partial}{\partial x_k}\left(a^\epsilon_{kl}(\epsilon)\frac{\partial u^\epsilon}{\partial x_l}\right) = f, \tag{1.30}$$

where $f \in L^2(\Omega)$ and $a^\epsilon_{kl}(\epsilon) := a_{kl}\left(\frac{x}{\epsilon}\right)$.

Homogenization of almost periodic media was first carried out by Kozlov [Koz78] using quasiperiodic approximations. Subsequently, an abstract approach was given in [OZ82, JKO94] which we now describe.

1.6.2 Cell Problem for Almost Periodic Media

We begin by introducing the cell problem for almost periodic operator $\mathcal{A}$. Consider the set $S = \{\nabla\phi : \phi \in \text{Trig}(\mathbb{R}^d;\mathbb{R})\}$ as a subset of $(B^2(\mathbb{R}^d))^d$, the Hilbert space of all d-tuples of $B^2(\mathbb{R}^d)$ functions. Let W denote the closure of S in $(B^2(\mathbb{R}^d))^d$. Let $U = (u_1, u_2, \ldots, u_d) \in W$ and $V = (v_1, v_2, \ldots, v_d) \in W$. On W, define the bilinear form

$$a(U,V) := \mathcal{M}(AU \cdot V). \tag{1.31}$$

Then clearly the bilinear form a is continuous and coercive on W. Let $\xi \in \mathbb{R}^d$. Define a linear form on W by

$$l_\xi(V) := -\mathcal{M}(A\xi \cdot V), \tag{1.32}$$

for $V \in W$. The linear form l_ξ is continuous on W. As a consequence, by Lax-Milgram lemma, the problem

$$a(N^\xi, V) = l_\xi(V), \ \forall V \in W \tag{1.33}$$

has a solution $N^\xi \in W$.

1.6.3 Homogenization of almost periodic operators

Suppose that u^ϵ converges weakly to a limit $u^* \in H^1(\Omega)$. Then, u^* satisfies

$$\mathcal{A}^* u := -\frac{\partial}{\partial x_k}\left(q^*_{kl}(x)\frac{\partial u}{\partial x_l}\right) = f, \tag{1.34}$$

where the homogenized coefficients q^*_{kl} for $\mathcal{A}^\epsilon$ are given by

$$q^*_{kl} = \mathcal{M}\left(e_k \cdot Ae_l + e_k \cdot AN^{e_l}\right), \tag{1.35}$$

where e_i denotes the unit vector in $\mathbb{R}^d$ with 1 in the i^{th} place and 0 elsewhere. A proof of this result may be found in [OZ82, JKO94]. However, the existence of almost periodic correctors is a difficult problem and only gradient of the corrector N^ξ is found in this abstract approach. Kozlov [Koz78] has proved the existence of correctors under the assumption of small divisors condition and used them to prove a rate of convergence result. Often, a penalized version of the cell problem is employed to obtain convergence rates [She15, SZ18]. Recently, Shen and his collaborators have proved the existence of bounded correctors for a wide class of almost periodic media using a quantification of almost periodicity [AGK16, AS16].

1.7 Our contributions

In our work, we recognized two difficulties associated with Bloch wave method:

1. **Regularity of Spectral Edges:** Many applications of Bloch waves require regularity of spectral edges, i.e., it requires that spectral edges be simple, isolated and non-degenerate. These properties may not be available for lowest edges of systems of partial differential operators, such as elasticity system where the bottom spectral edge has multiplicity 3, and for internal edges

of scalar operators [Kuc16]. At lowest edges, these difficulties have been circumvented by the use of one-parameter perturbation theory [Kat95] by Birman-Suslina [BS04] and Sivaji Ganesh-Vanninathan [SGV05], etc. It is not clear whether directional regularity can be applied to the case of internal edges.

2. **Structural Restriction:** The direct integral decomposition, on which Bloch wave method is based, is restricted to periodic operators. Although certain spectral tools seem to exist in the noncommutative geometry literature [BT81], this apparatus is known to be difficult and it is not clear whether it will lead to new computational techniques and quantitative results. However, given the widespread use of Bloch wave methods in physics literature, Tartar [Tar09] feels it may be extended to aperiodic media. Such a goal is indeed attainable as has been shown by the extension of the Bloch wave method to Hashin-Shtrikman structures [BCG+16, BCG+18].

The present thesis develops the Bloch wave method in these directions. In particular, we have worked on the spectral theory of periodic differential operators in order to develop tools that would allow us to establish homogenization results in the presence of multiple spectral edges. Further, in order to move beyond periodicity, we have used periodic approximations to almost-periodic operators in Bloch wave method to obtain homogenization results. This has also allowed us to explore approximations of homogenized tensors of almost periodic media and establish certain rate of convergences for these approximations. We will now explain our contributions in more detail. The results of Subsections 1.7.1, 1.7.2, 1.7.3 may be found in the preprint [1] which has been accepted for publication in *Asymptotic Analysis*. The results of Subsection 1.7.4 may be found in the arXiv preprint [2]. The results of Subsection 1.7.5 may be found in the arXiv preprint [3].

1.7.1 Genericity results on Bloch spectrum

In Section 1.3, we recalled some genericity results about spectrum of compact operators. In this thesis, we generalize Albert's result to the spectrum of periodic operators in two ways.

Theorem A. *Let* $\eta_0 \in Y'$. *The eigenvalues of the shifted operator* (1.10) $\mathcal{A}(\eta_0)$ *are generically simple with respect to the space of all coefficient matrices* A *that are measurable, bounded and positive definite.*

In the theorem above, we overcome the following difficulties:

1. Theorem A is an extension of the theorem of Albert [Alb75] where the potential V is the quantity of interest. For applications in the theory of homogenization, the periodic matrix A in the divergence type elliptic operator $-\nabla \cdot (A\nabla)$ is of physical importance. Thus, the perturbation is sought in the second-order term as opposed to the zeroth-order term in [Alb75].

2. Albert's result is applicable to operators with discrete spectrum, whereas a periodic operator typically has no eigenvalues. However, the spectrum may be analyzed through Bloch eigenvalues which introduces an extra parameter $\eta \in Y'$ to the problem. Hence, the method of Albert is applied in a fiberwise manner.

3. The fiber $\mathcal{A}(\eta)$ is an operator with complex-valued coefficients. The determination of real-valued perturbation for the shifted operator $\mathcal{A}(\eta)$, which has complex-valued coefficients, poses additional difficulties, when coupled with the lack of regularity of the coefficients which the applications demand.

Further, by applying fiberwise perturbation on the fibered operator $\int_{Y'}^{\oplus} \mathcal{A}(\eta)\,d\eta$, we can make sure that the corresponding eigenvalue of interest is simple for all parameter values.

Theorem B. *Let* $m \in \mathbb{N}$. *Let* $\lambda_m(\eta)$ *be the* m^{th} *Bloch eigenvalue of the operator* $\mathcal{A} = -\nabla \cdot (A\nabla)$, *where* A *satisfies (A1), (A2), (A3) and the entries of* A *are periodic. There exists an operator* $\tilde{\mathcal{A}}$, *unitarily equivalent to the operator* $\int_{Y'}^{\oplus} (\mathcal{A}(\eta) + \mathcal{B}(\eta))\,d\eta$, *such that the perturbed eigenvalue* $\tilde{\lambda}_m(\eta)$ *is simple for all* $\eta \in Y'$. *The perturbation* $\int_{Y'}^{\oplus} \mathcal{B}(\eta)\,d\eta$ *has the form* $\mathcal{B}(\eta) = -(\nabla + i\eta) \cdot (B(\eta)(\nabla + i\eta))$, *where the coefficients* $B(\eta) \in L^\infty_\sharp(Y)$ *are piecewise constant in* Y' *and may be chosen as small as desired in the* L^∞*-norm.*

While Theorem B achieves global simplicity for a Bloch eigenvalue, the perturbed operator is no longer a differential operator, i.e., it is non-local.

1.7.2 Simplicity of spectral edges

In Section 1.4, we reviewed the literature about regularity of spectral edges. We prove the following theorem concerning simplicity of spectral edges.

Theorem C. *Let the matrix* A *satisfy (A1), (A2), (A3). Also, assume that the entries of* A *are periodic. Let* λ_0 *correspond to the upper edge of a spectral* *gap of $\mathcal{A}$ and let m be the smallest index such that the Bloch eigenvalue λ_m attains λ_0. Suppose that either*

(i) the entries of A belong to the class $W^{1,\infty}_\sharp(Y,\mathbb{R})$, or

(ii) the spectral edge λ_0 is attained by $\lambda_m(\eta)$ at finitely many points.

Then, there exists a matrix B with $L^\infty_\sharp(Y,\mathbb{R})$-entries and $t_0 > 0$ such that for every $t \in (0, t_0]$, a spectral edge is achieved by the m^{th} Bloch eigenvalue of the operator $\tilde{\mathcal{A}} = -\nabla \cdot (A + tB)\nabla$ and the spectral edge is simple. An analogous statement holds for the lower edge of the spectral gap.

Our theorem differs from previous works in the following respects. Simplicity of spectral edge under perturbation has been proved for Schrödinger operators by Klopp and Ralston [KR00]. These perturbations are applied in the lowest order term whereas our perturbation is in the second order. Klopp and Ralston make use of De Giorgi-Nash-Moser regularity for the Bloch eigenfunction in an essential way. The regularity theorem does not extend to derivatives of the eigenfunctions. As a consequence, the proof of Klopp and Ralston could only be successfully applied to second order perturbations provided we had more regularity for the coefficients, $W^{1,\infty}$ or $C^{0,\alpha}$. However, this excess regularity is not suitable for applications in the theory of homogenization where coefficients are typically only measurable and bounded. We are able to relax the regularity requirement to L^∞ coefficients by imposing an extra hypothesis that the spectral edge is attained at finitely many points. Recall that this is one of the defining conditions of regularity of spectral edge. Therefore our theorem illustrates the importance of regularity.

1.7.3 Applications to internal edge homogenization

We also prove a theorem on norm-resolvent convergence estimates in $L^2(\mathbb{R}^d)$ for homogenization of $\mathcal{A}^\epsilon$ at an internal (i.e. non-zero) multiple spectral edge. We will see that this estimate depends on the shape and structure of the spectral edge which is typically assumed to be regular [BS06]. We impose some assumptions on shape and structure of a multiple spectral edge. This theorem illustrates a situation where perturbation theory may be applied in order to obtain homogenization results. The shape and structure of an internal edge is largely unknown and it is expected to be known only through experiments unlike the case of the lowest edge which is known to be regular.

1.7.4 Bloch approach to almost periodic homogenization

Bloch wave homogenization through periodic approximations

The assumption of periodicity in homogenization has many advantages: a variety of tools are available, rates of convergence are easy to obtain, periodic media is easier to design. However, non-periodic media occurs in nature, such as glass, quasicrystals, and many physical effects may not be explained solely through periodicity. Bloch wave method relies on direct integral decomposition of periodic operators. For almost periodic operators, a suitable direct integral decomposition is not available. To overcome this difficulty, we make use of periodic approximations, which are defined by a "restrict and periodize" operation, employed earlier by Bourgeat and Piatnitski [BP04] for random homogenization. Further, the approximations introduce varying Hilbert spaces to the problem, which are handled by working in the Besicovitch space of almost periodic functions.

First Step: The first step in our procedure is the approximation of almost periodic functions using periodic functions. This is done by the "restrict and periodize" procedure as developed in [BP04]. We propose periodic approximations of the operator

$$\mathcal{A}u := -\mathrm{div}(A\nabla u) = -\frac{\partial}{\partial y_k}\left(a_{kl}(y)\frac{\partial u}{\partial y_l}\right), \tag{1.36}$$

where the entries of A are Bohr almost periodic. For $R > 0$, we denote by A^R the periodic approximation of A at level R, defined as $A^R = (a^R_{kl}(y)) = (a_{kl}(y))$ for $y \in Y_R := [0, 2\pi R)^d$, and $a^R_{kl}(y + 2\pi R p) = a_{kl}(y)$ for $p \in \mathbb{Z}^d$. The following operator will serve as the periodic approximation to $\mathcal{A}$.

$$\mathcal{A}^R u := -\mathrm{div}(A^R\nabla u) = -\frac{\partial}{\partial y_k}\left(a^R_{kl}(y)\frac{\partial u}{\partial y_l}\right). \tag{1.37}$$

Second Step: The second step is to develop the Bloch wave apparatus for (1.37). Let $\lambda^R_n(\eta), \phi^R_n(\eta)$ denote the Bloch eigenvalue and eigenvectors of the operator (1.37). Then, the homogenized tensor $a^{R,*}_{kl}$ for the operator

$$\mathcal{A}^{R,\epsilon} u := -\mathrm{div}(A^{R,\epsilon}\nabla u) = -\frac{\partial}{\partial x_k}\left(a^R_{kl}(x/\epsilon)\frac{\partial u}{\partial x_l}\right). \tag{1.38}$$

is given by

$$a^{R,*}_{kl} = \frac{1}{2}\frac{\partial^2 \lambda^R_1}{\partial \eta_k \partial \eta_l}(0). \tag{1.39}$$

Third Step: The third step is to achieve homogenization for the equation

$$\mathcal{A}^{\epsilon} u^{\epsilon} := -\mathrm{div}(A^{\epsilon}\nabla u^{\epsilon}) = -\frac{\partial}{\partial x_k}\left(a_{kl}(x/\epsilon)\frac{\partial u^{\epsilon}}{\partial x_l}\right) = f. \tag{1.40}$$

This task is performed by first writing equation (1.40) in Bloch space. Let ψ_0 be a fixed element in $\mathcal{D}(\Omega)$ with support K. Since u^ϵ satisfies $\mathcal{A}^\epsilon u^\epsilon = f$, $\psi_0 u^\epsilon$ satisfies

$$\mathcal{A}^{R,\epsilon}(\psi_0 u^\epsilon)(x) = \psi_0 f(x) + g^\epsilon(x) + h^\epsilon(x) + l^{R,\epsilon}(x) \text{ in } \mathbb{R}^d, \tag{1.41}$$

where

$$g^\epsilon(x) := -\frac{\partial\psi_0}{\partial x_k}(x)a^\epsilon_{kl}(x)\frac{\partial u^\epsilon}{\partial x_l}(x), h^\epsilon(x) := -\frac{\partial}{\partial x_k}\left(\frac{\partial\psi_0}{\partial x_l}(x)a^{R,\epsilon}_{kl}(x)u^\epsilon(x)\right),$$
$$l^{R,\epsilon}(x) := -\frac{\partial}{\partial x_k}\left(\psi_0(x)\left(a^{R,\epsilon}_{kl}(x) - a^\epsilon_{kl}(x)\right)\frac{\partial u^\epsilon}{\partial x_l}(x)\right).$$

Bloch wave homogenization is achieved by passing to the limit in equation (1.41) first as ϵ goes to 0, followed by $R \to \infty$.

An interesting result that we have proved is a *module containment* result for the approximate correctors. An abstract approach to the cell problem for almost periodic media was given in [JKO94] which was described in Section 1.6. Therefore, for periodic media (which forms a subclass of almost periodic media) there are two possible variational formulations (1.9), and (1.33). It is natural to ask whether these two cell problems are consistent. We prove this through the module containment result, which may be paraphrased to say that the frequencies of the correctors are generated from the frequencies of the coefficients of the operator.

Approximations of homogenized tensor

It was proved in [BP04] that the approximate homogenized tensors $a^{R,*}_{kl}$ (1.39) converge to the homogenized tensor q^*_{kl} (1.35) of almost periodic operator. Although we are unable to provide a convergence rate for the approximations through periodizations, we prove a rate of convergence result for Dirichlet approximations to the homogenized tensor.

The following is its Dirichlet approximation, which is the truncation of (1.35) on a cube $Y_R = [-\pi R, \pi R)^d$ of side length $2\pi R$. Given $v \in \mathbb{R}^d$, find $w^{R,D,v} \in H^1_0(Y_R)$ such that

$$-\nabla \cdot A(v + \nabla w^{R,D,v}) = 0. \tag{1.42}$$

Then Dirichlet approximation $A^{R,D,*} = \left(a^{R,D,*}_{kl}\right)$ to the homogenized tensor is defined by

$$a^{R,D,*}_{kl} := \mathcal{M}_{Y_R}\left(a_{kl} + \sum_{j=1}^{d} a_{kj}\frac{\partial w^{R,D,e_l}}{\partial y_j}\right). \tag{1.43}$$

For a matrix A with entries in $AP(\mathbb{R}^d)$, define the following modulus of almost periodicity:

$$\rho(A, L) := \sup_{y\in\mathbb{R}^d} \inf_{|z|\le L} ||A(\cdot + y) - A(\cdot + z)||_{L^\infty(\mathbb{R}^d)}. \tag{1.44}$$

It follows that A is almost periodic if and only if $\rho(A, L) \to 0$ as $L \to \infty$. We prove the following theorem on the rate of convergence.

Theorem D. *Let ρ satisfy $\rho(A, L) \lesssim 1/L^\tau$ for some $\tau > 0$. There exists a $\beta \in (0, 1)$ such that*

$$\left|q^*_{kl} - a^{R,D,*}_{kl}\right| \lesssim \frac{1}{R^\beta}. \tag{1.45}$$

We point out the following features of the theorem above:

1. For the proof of Theorem D, we follow the strategy in [BP04] where similar convergence rates were obtained for stochastic media satisfying a strong mixing condition. In their proof, Bourgeat and Piatnitski [BP04] employ penalized/regularized cell problems which makes the analysis possible.

2. In general, almost periodic media is known to be ergodic but not mixing [Sim82]. Therefore, a quantification of almost periodic media was required. This is provided in the form of $\rho(A)$ in the paper of Shen [She15]. Previously, convergence rate results were available for quasiperiodic media under the small divisors condition. Shen [She15] proves that the small divisors condition implies a polynomial decay of $\rho(A; L)$ which is the hypothesis in Theorem D.

3. Convergence rates for periodizations appears to be a difficult problem.

1.7.5 Bloch wave homogenization of quasiperiodic media

Quasiperiodic functions form a subclass of almost periodic functions. In general, quasiperiodic functions may be lifted to periodic functions on a higher dimensional space. This suggests a second approach to Bloch wave homogenization of quasiperiodic media. Let A be a $d \times d$ matrix whose entries are quasiperiodic functions in $\mathbb{R}^d$, then there exists a $d \times d$ periodic matrix B and a $M \times d$ matrix Λ such that $A(x) = B(\Lambda x)$ where $M > d$. Under the transformation $x \mapsto \Lambda x$, the operator $\mathcal{A} = -\text{div}(A\nabla)$ in $\mathbb{R}^d$ with quasiperiodic coefficients may be lifted to the following periodic operator in $\mathbb{R}^M$.

$$\mathcal{C} = -\Lambda^T\nabla_y \cdot B(y)\Lambda^T\nabla_y\,. \tag{1.46}$$

The technique of lifting of quasiperiodic operators has been employed in [Koz78, GH16, BLBL15, WGC18]. The main difficulty in obtaining Bloch waves for $\mathcal{C}$ is that $\mathcal{C}$ is a degenerate operator. This is circumvented by regularizing the Bloch spectral problem. The method of regularization was suggested in [BLBL15] for the degenerate cell problem.

Let $Q = [0, 2\pi)^M$ and let $\mathcal{C}(\eta) := -(\Lambda^T\nabla_y + i\eta) \cdot B(y)(\Lambda^T\nabla_y + i\eta)$ denote the shifted operator associated to $\mathcal{C}$ defined for $\eta \in Y' := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$. We define the regularized version of shifted operator as

$$\mathcal{C}^\delta(\eta) := -(\Lambda^T\nabla_y + i\eta) \cdot B(y)(\Lambda^T\nabla_y + i\eta) + \delta\Delta.$$

The eigenfunctions of the regularized operator $\mathcal{C}^\delta(\eta)$ with periodic boundary conditions serve as approximate Bloch waves. The homogenized tensor for quasiperiodic media is characterized in terms of the limit of the Hessian of the first approximate Bloch eigenvalue as the regularization parameter δ tends to zero. Moreover, the restriction of the approximate Bloch eigenfunctions to $\mathbb{R}^d$ is quasiperiodic. This allows us to define a quasiperiodic Bloch transform which is employed for obtaining homogenization limit of quasiperiodic media.

This method suggests another method for Bloch wave homogenization of almost periodic operators through approximation of almost periodic functions by trigonometric polynomials. In Section 1.7.4, the media is approximated by its periodization on a large cube. In the present method, the difficulty of working on a large domain is traded for the difficulty of working in high dimensions.

Plan of thesis

In Chapter 2, we will develop genericity results in spectral theory of periodic elliptic operators. In Chapter 3, we establish the theorem on internal edge homogenization in the presence of multiplicity. In Chapter 4, we present a Bloch approach to almost periodic homogenization. In Chapter 5, we prove a rate of convergence result for approximations of homogenized tensor of almost periodic media which are supplemented with numerical examples. In Chapter 6, we shall prove homogenization theorem for quasiperiodic media using Bloch wave method.

Chapter 2

Simplicity of Spectral Edges

We consider the spectrum of a second-order elliptic operator in divergence form with periodic coefficients, which is known to be completely described by Bloch eigenvalues. We show that under small perturbations of the coefficients, a multiple Bloch eigenvalue can be made simple. The Bloch wave method of homogenization relies on the regularity of spectral edge. The spectral tools that we develop, allow us to obtain simplicity of an internal spectral edge through perturbation of the coefficients.

2.1 Introduction

An introduction to the spectral theory of periodic elliptic operators is provided in Section 1.2. To recapitulate, the spectrum of a periodic elliptic operator is a union of intervals whose endpoints are called as spectral edges. Further, the spectrum is fully described by Bloch eigenvalues whose behaviour near spectral edges determine a variety of physical phenomena. Simplicity of spectral edges is a useful property for determining homogenized coefficients. In this chapter, we make a small perturbation in the coefficients of a divergence-type periodic elliptic operator so that the resulting operator has a desired simple spectral edge. The main tool is the perturbation theory of Kato and Rellich [Kat95]. The contents of this chapter form a section of the paper [1] which will appear in the journal *Asymptotic Analysis*.

2.2 Main Results

Let $\mathcal{S}ym(d)$ denote the space of all real symmetric matrices, i.e., if $A = (a_{kl}) \in \mathcal{S}ym(d)$, then $a_{kl} = a_{lk}$. Let

$$M_B^{>} = \{A : \mathbb{R}^d \to \mathcal{S}ym(d) : a_{kl} \in L^\infty_\sharp(Y, \mathbb{R}) \text{ and } A \text{ is coercive } \}.$$

$M_B^>$ may be identified as a subset of the space of $d(d+1)/2$-tuples of $L_\sharp^\infty$ functions and we shall use the norm-topology on this space in our further discussion. A Baire space is a topological space in which the countable intersection of dense open sets is dense. Note that $M_B^>$ is an open subset of the space of all symmetric matrices with $L_\sharp^\infty(Y,\mathbb{R})$ entries, which forms a complete metric space, and hence $M_B^>$ is a Baire Space. We shall call a property *generic* in a topological space X, if it holds on a set whose complement is of first category in X. In particular, a property that is generic on a Baire space holds on a dense set.

The rest of the section will be devoted to the statements of the main results.

Theorem 2.1. *Let $\eta_0 \in Y'$. The eigenvalues of the shifted operator $\mathcal{A}(\eta_0)$ are generically simple with respect to the coefficients $A = (a_{kl})_{k,l=1}^d$ in $M_B^>$.*

Remark 2.2. Theorem 2.1 is an extension of the theorem of Albert [Alb75] which proves that the eigenvalues of $-\Delta + V$ are generically simple with respect to $V \in C^\infty(M)$ for a compact manifold M. The potential V is the quantity of interest for Schrödinger operator, $-\Delta + V$. For the applications that we have in mind, for example, the theory of homogenization, the periodic matrix A in the divergence type elliptic operator $-\nabla \cdot (A\nabla)$ is of physical importance. The spectrum of such operators is not discrete, and is analyzed through Bloch eigenvalues, which introduces an extra parameter $\eta \in Y'$ to the problem. The determination of real-valued perturbation for the shifted operator $\mathcal{A}(\eta)$, which has complex-valued coefficients, poses additional difficulties, when coupled with the lack of regularity of the coefficients which the applications demand.

The operator $\mathcal{A}$ in $L^2(\mathbb{R}^d)$ is unitarily equivalent to the fibered operator $\int_{Y'}^\oplus \mathcal{A}(\eta)\,d\eta$ in the Bochner space $L^2(Y', L_\sharp^2(Y))$. In the next theorem, we apply a fiberwise perturbation to the direct integral decomposition of $\mathcal{A}$ to obtain an operator with a simple Bloch eigenvalue.

Theorem 2.3. *Let $m \in \mathbb{N}$. Let $\lambda_m(\eta)$ be the m^{th} Bloch eigenvalue of the operator $\mathcal{A} = -\nabla \cdot (A\nabla)$, where $A \in M_B^>$. There exists an operator $\tilde{\mathcal{A}}$, unitarily equivalent to the operator $\int_{Y'}^\oplus \mathcal{A}(\eta) + \mathcal{B}(\eta)\,d\eta$, such that the perturbed eigenvalue $\tilde{\lambda}_m(\eta)$ is simple for all $\eta \in Y'$. The perturbation $\int_{Y'}^\oplus \mathcal{B}(\eta)\,d\eta$ has the form $\mathcal{B}(\eta) = -(\nabla + i\eta) \cdot (B(\eta)(\nabla + i\eta))$, where the coefficients $B(\eta) \in L_\sharp^\infty(Y)$ are piecewise constant in Y' and may be chosen as small as desired in L^∞-norm.*

A spectral edge λ_0 is said to be *simple* if the set $\{m \in \mathbb{N} : \exists \eta \in Y'$ such that $\lambda_m(\eta) = \lambda_0\}$ is a singleton. A spectral edge is said to be multiple if it is not simple.

Theorem 2.4. *Let* $A \in M_B^>$. *Further, suppose that its entries* $A = (a_{kl})_{k,l=1}^d$ *belong to the class* $W_\sharp^{1,\infty}(Y, \mathbb{R})$. *Let* λ_0 *correspond to an edge of a spectral gap of* $\mathcal{A}$ *and let* m *be the smallest index such that the Bloch eigenvalue* λ_m *attains* λ_0. *Then there exists a matrix* $B = (b_{kl})_{k,l=1}^d$ *with* $C_\sharp^\infty(Y, \mathbb{R})$*-entries and* $t_0 > 0$ *such that for every* $t \in (0, t_0]$, *a spectral edge is achieved by the Bloch eigenvalue* $\lambda_m(\eta; A + tB)$ *of the operator* $\tilde{\mathcal{A}} = -\nabla \cdot (A + tB)\nabla$ *and the spectral edge is simple.*

Theorem 2.5. *Let* $A \in M_B^>$. *Let* λ_0 *correspond to an edge of a spectral gap of* $\mathcal{A}$ *and let* m *be the smallest index such that the Bloch eigenvalue* λ_m *attains* λ_0. *Assume that the spectral edge is attained by* $\lambda_m(\eta)$ *at finitely many points. Then there exists a matrix* $B = (b_{kl})_{k,l=1}^d$ *with* $L_\sharp^\infty(Y, \mathbb{R})$*-entries and* $t_0 > 0$ *such that for every* $t \in (0, t_0]$, *a spectral edge is achieved by the Bloch eigenvalue* $\lambda_m(\eta; A+tB)$ *of the operator* $\tilde{\mathcal{A}} = -\nabla \cdot (A+tB)\nabla$ *and the spectral edge is simple.*

Remark 2.6.

1. While Theorem 2.3 achieves global simplicity for a Bloch eigenvalue, the perturbed operator is no longer a differential operator, i.e., it is non-local. In the theory of homogenization, non-local terms usually appear as limits of non-uniformly bounded operators [Bri02b], [Bri02a]. In the presence of crossing modes, non-locality appears in the theory of effective mass [MCFK].

2. Theorem 2.4 is an adaptation of the theorem of Klopp and Ralston [KR00] to divergence-type operators. Their proof relies heavily on the Hölder regularity for weak solutions of divergence-type operators. In our proof, we require Hölder continuity of the solutions as well as their derivatives. Hence, we have to impose $W^{1,\infty}$ condition on the coefficients.

3. In Theorem 2.5, we weaken the $W^{1,\infty}$ requirement on the coefficients under assumption of finiteness on the number of points at which the spectral edge is attained. This is essential for the applications that we have in mind, in the theory of homogenization, where only L^∞ regularity is available on the coefficients.

4. Bloch wave method belongs to the family of multiplier techniques in partial differential equations. In particular, exponential type multipliers, $e^{\tau\phi}$, with real exponents, are used in obtaining Carleman estimates for elliptic operators [Rou12].

5. Any operator of the form $-\nabla \cdot A\nabla$ in $L^2(\mathbb{R}^d)$ may be written in direct integral form, provided A is periodic. A satisfactory spectral theory for such operators is available for real symmetric A. However, non-selfadjoint operators are becoming increasingly important in physics [Sjo10]. For non-symmetric A, the eigenvalues of the fibers $\mathcal{A}(\eta)$ may no longer be real and the eigenfunctions may not form a complete set. These difficulties were surmounted in proving the Bloch wave homogenization theorem for non-selfadjoint operators in [SGV04]. Nevertheless, the generalized eigenfunctions form a complete set for a large class of elliptic operators of even order [Agm62]. However, we are not aware of physical interpretations of complex-valued Bloch-type eigenvalues.

6. The results of this chapter would have similar analogues for internal edges of an elliptic system of equations, for example, the elasticity system. It would be interesting to consider these problems for the spectrum of non-elliptic operators such as the Maxwell operator.

The plan of this chapter is as follows; in Section 2.3, we prove Theorem 2.1 on generic simplicity of Bloch eigenvalues at a point. In Section 2.4, we prove Theorem 2.3 and In Sections 2.5 and 2.6, we prove Theorems 2.4 and 2.5 concerning generic simplicity of spectral edges.

2.3 Local Simplicity of Bloch eigenvalues

Let $\eta_0 \in Y'$. Let P be the set defined by

$$P := \{A \in M_B^> : \text{ the eigenvalues of } \mathcal{A}(\eta_0) \text{ are simple}\}.$$

We can write the set P as an intersection of countably many sets as follows: Let $P_0 := M_B^>$, and

$$\begin{aligned} P_n &:= \{A \in M_B^> : \text{ the first } n \text{ eigenvalues of } \mathcal{A}(\eta_0) \text{ are simple}\} \\ &= \{A \in M_B^> : \lambda_1(\eta_0) < \ldots < \lambda_n(\eta_0) < \lambda_{n+1}(\eta_0) \leq \lambda_{n+2}(\eta_0) \leq \ldots\}. \end{aligned}$$

Note that

$$P \subseteq \ldots \subseteq P_n \subseteq P_{n-1} \subseteq \ldots \subseteq P_1 \subseteq P_0 \qquad \text{and} \qquad P = \bigcap_{n=0}^{\infty} P_n.$$

We shall require the following two lemmas.

Lemma 2.7. P_n *is open in* $M_B^>$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Lemma 2.8. P_{n+1} *is dense in* P_n *for all* $n \in \mathbb{N} \cup \{0\}$.

Proof. (of Theorem 2.1) We recall that a property is said to be generic in a topological space X, if it holds on a set whose complement is of first category in X. We can write P as the countable intersection $P = \bigcap_{n=0}^{\infty} P_n$, where P_n is an open and dense set in $M_B^>$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, the complement of P is a set of first category. Therefore, the simplicity of eigenvalues of $\mathcal{A}(\eta_0)$ is a generic property in $M_B^>$. □

The rest of this section is devoted to the proofs of Lemmas 2.7 and 2.8.

2.3.1 Proof of Lemma 2.7

The proof of Lemma 2.7 requires continuous dependence of eigenvalues of the shifted operator $\mathcal{A}(\eta)$ on its coefficients. This can be proved using the Courant-Fischer min-max principle, which states that

$$\lambda_m(\eta_0) = \min_{\dim F = m} \max_{v \in F\setminus\{0\}} \frac{\int_Y A(\nabla + i\eta_0)v.\overline{(\nabla + i\eta_0)v}\ dx}{\int_Y |v|^2\ dx},$$

where F ranges over all subspaces of $H^1_\sharp(Y)$ of dimension m.

Proposition 2.9. *Let* $A_1, A_2 \in M_B^>$ *and let* $\eta \mapsto \lambda^1_n(\eta), \eta \mapsto \lambda^2_n(\eta)$ *be the n-th Bloch eigenvalues of the operators* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *respectively. Then*

$$|\lambda^1_n(\eta_0) - \lambda^2_n(\eta_0)| \le c_n(\eta_0)||A_1 - A_2||_{L^\infty},$$

where $c_n(\eta_0)$ *is the* n^{th} *eigenvalue of the shifted Laplacian* $H(\eta_0) := -(\nabla + i\eta_0)^2$ *on* Y *with periodic boundary conditions.* □

Proof. Let $a_1(v) = \int_Y A_1(\nabla + i\eta_0)v.\overline{(\nabla + i\eta_0)v}\ dy$ and $a_2(v) = \int_Y A_2(\nabla + i\eta_0)v.\overline{(\nabla + i\eta_0)v}\ dy$ be the quadratic forms that appear in the min-max principle. We have

$$\begin{aligned} |a_1(v) - a_2(v)| &= \left|\int_Y (A_1 - A_2)(\nabla + i\eta_0)v.\overline{(\nabla + i\eta_0)v}\ dy\right| \\ &\le ||A_1 - A_2||_{L^\infty} \int_Y |(\nabla + i\eta_0)v|^2\ dy, \end{aligned}$$

Therefore,

$$a_1(v) \leq a_2(v) + \|A_1 - A_2\|_{L^\infty} \int_Y |(\nabla + i\eta_0)v|^2 \, dy.$$

Now, divide both sides by $\int_Y |v|^2 \, dy$, the $L^2_\sharp(Y)$ inner product of v with itself and apply the appropriate min-max to obtain

$$\lambda^1_m(\eta_0) \leq \lambda^2_m(\eta_0) + c_m(\eta_0)\|A_1 - A_2\|_{L^\infty}.$$

Notice that the constant $c_m(\eta_0)$ is precisely the m^{th} eigenvalue of the shifted Laplacian $H(\eta_0)$ on Y with periodic boundary conditions. By interchanging the role of A_1 and A_2, the inequality

$$\lambda^2_m(\eta_0) \leq \lambda^1_m(\eta_0) + c_m(\eta_0)\|A_2 - A_1\|_{L^\infty},$$

is obtained, which completes the proof of this proposition. □

Remark 2.10. In [CV97], the Bloch eigenvalues have been proved to be Lipschitz continuous in $\eta \in Y'$. Indeed, one may prove that the Bloch eigenvalues are jointly continuous in $\eta \in Y'$ and the coefficients of the operator.

Proof of Lemma 2.7. Let $A \in P_n$ and

$$\delta = \min\{\lambda_{j+1}(\eta_0) - \lambda_j(\eta_0) : j = 1, 2, \ldots, n\}.$$

Let $c = \max\limits_{1\leq j\leq n+1} c_j(\eta_0)$, where $c_j(\eta_0)$ is the j^{th} eigenvalue of the shifted Laplacian $H(\eta_0)$ on Y with periodic boundary conditions.

Let

$$U = \left\{ A' \in M^>_B : \|A - A'\|_{L^\infty} < \frac{\delta}{4c} \right\}.$$

Then U is an open set in $M^>_B$ containing A. We shall show that U is a subset of P_n. Let $A' \in U$. Let $\{\lambda'_j(\eta),\ j = 1, 2, \ldots\}$ be the Bloch eigenvalues of operator $\mathcal{A}'$ associated to A'. For $j = 1, 2, \ldots, n+1$, we have:

$$|\lambda'_j(\eta_0) - \lambda_j(\eta_0)| \leq c_j(\eta_0)\|A - A'\|_{L^\infty} \leq c_j(\eta_0)\frac{\delta}{4c} \leq \frac{\delta}{4}.$$

Hence,

$$\begin{aligned}
\delta &\leq \lambda_{j+1}(\eta_0) - \lambda_j(\eta_0) \\
&\leq |\lambda'_{j+1}(\eta_0) - \lambda_{j+1}(\eta_0)| + |\lambda'_j(\eta_0) - \lambda'_{j+1}(\eta_0)| + |\lambda'_j(\eta_0) - \lambda_j(\eta_0)| \\
&\leq \frac{\delta}{4} + |\lambda'_j(\eta_0) - \lambda'_{j+1}(\eta_0)| + \frac{\delta}{4} \\
&= \frac{\delta}{2} + \lambda'_{j+1}(\eta_0) - \lambda'_j(\eta_0).
\end{aligned}$$

Therefore, $\lambda'_{j+1}(\eta_0) - \lambda'_j(\eta_0) \geq \frac{\delta}{2} > 0$ for $j = 1, 2, \ldots, n$. Therefore, the first n Bloch eigenvalues of $\mathcal{A}'$ are simple at η_0, as required. □

2.3.2 Proof of Lemma 2.8

In this section, we shall use perturbation theory of selfadjoint operators to prove Lemma 2.8. Let $A \in M_B^>$ and B be a symmetric matrix with $L^\infty_\sharp(Y, \mathbb{R})$-entries. For $|\tau| < \sigma_0 := \frac{\alpha}{2\|B\|_{L^\infty}}$, $A + \tau B \in M_B^>$, where α is a coercivity constant for A as in (A3). Consider the operator $\mathcal{A}(\eta_0) + \tau\mathcal{B}(\eta_0)$ in $L^2_\sharp(Y)$. We shall prove in Appendix A that the operator family $\mathcal{F}(\tau) = \mathcal{A}(\eta_0) + \tau\mathcal{B}(\eta_0)$ is a selfadjoint holomorphic family of type (B) for $|\tau| < \sigma_0$. A similar verification is performed in [SGV04]. For the definition of selfadjoint holomorphic family of type (B) and related notions, see Kato [Kat95].

We shall make use of the following theorem which asserts the existence of a sequence of eigenpairs associated with a selfadjoint holomorphic family of type (B), analytic in $\tau \in (-\sigma_0, \sigma_0)$. The proof of this theorem dates back to Rellich, hence we shall call these eigenvalue branches as Rellich branches.

Theorem 2.11. *(Kato-Rellich) Let $\sigma_0 > 0$. Let $\mathcal{F}(\tau)$ be a selfadjoint holomorphic family of type (B), defined for $\tau \in R$, where $R = \{z \in \mathbb{C} : |\mathrm{Re}(z)| < \sigma_0, |\mathrm{Im}(z)| < \sigma_0\}$. Let $\mathcal{F}(\tau) + C_* I$ have compact resolvent for some $C_* \in \mathbb{R}$. Then there exists a sequence of scalar-valued functions $(\lambda_j(\tau))_{j=1}^\infty$ and $L^2_\sharp(Y)$-valued functions $(u_j(\tau))_{j=1}^\infty$ defined on $I = (-\sigma_0, \sigma_0)$, such that*

1. *For each fixed $\tau \in I$, the sequence $(\lambda_j(\tau))_{j=1}^\infty$ represents all the eigenvalues of $\mathcal{F}(\tau)$ counting multiplicities and the functions $(u_j(\tau))_{j=1}^\infty$ represent the corresponding eigenvectors.*

2. *For each $j \in \mathbb{N}$, the functions $(\lambda_j(\tau))_{j=1}^\infty$ and $(u_j(\tau))_{j=1}^\infty$ are real-analytic on I with values in $\mathbb{R}$ and $L^2_\sharp(Y)$ respectively.*

3. *The sequence $(u_j(\tau))_{j=1}^\infty$ is orthonormal in $L^2_\sharp(Y)$.*

4. *Suppose that the m^{th} eigenvalue of $\mathcal{F}(\tau)$ at $\tau = 0$ has multiplicity p, i.e.,*

$$\lambda_m(0) = \lambda_{m+1}(0) = \ldots = \lambda_{p+m-2}(0) = \lambda_{p+m-1}(0).$$

Let $K \subset \mathbb{R}$ be an interval, with $\overline{K}$ containing the eigenvalue $\lambda_m(0)$ and no other eigenvalue. Then for $|\tau|$ sufficiently small, $\lambda_m(\tau)$, $\lambda_{m+1}(\tau), \ldots, \lambda_{p+m-1}(\tau)$ are the only eigenvalues of $\mathcal{F}(\tau)$, counting multiplicities, lying in the interval K.

By Kato-Rellich Theorem, an eigenvalue $\lambda(\eta_0)$ of $\mathcal{F}(0)$ of multiplicity h, splits into h real-analytic functions $(\lambda_m(\tau; \eta_0))_{m=1}^h$. Further, the corresponding eigenfunctions $(u_m(\tau;\eta_0))_{m=1}^h$ are also real-analytic. Thus, for $m = 1, 2, \ldots, h$ and $\tau \in (-\sigma_0, \sigma_0)$, we may write:

$$\lambda_m(\tau;\eta_0) = \lambda(\eta_0) + \tau a_m(\eta_0) + \tau^2 \beta_m(\tau, \eta_0),$$
$$u_m(\tau;\eta_0) = u_m(\eta_0) + \tau v_m(\eta_0) + \tau^2 w_m(\tau, \eta_0),$$

where $\beta_m(\tau, \eta_0)$ and $w_m(\tau, \eta_0)$ are real-analytic functions.

The proof of Lemma 2.8 will rely on the fact that we may choose B in such a way that $a_m(\eta_0) \neq a_n(\eta_0)$ for some $m, n \in \{1, 2, \ldots, h\}$. Then for sufficiently small τ, $\lambda_m(\tau;\eta_0) \neq \lambda_n(\tau;\eta_0)$. In that case, the multiplicity of the perturbed Bloch eigenvalue at η_0 will be less than h.

The eigenpairs satisfy the following equation:

$$(-(\nabla + i\eta_0) \cdot (A + \tau B)(\nabla + i\eta_0) - \lambda_m(\tau, \eta_0))\, u_m(\tau, \eta_0) = 0.$$

Differentiating the above with respect to τ and setting τ to 0, we obtain:

$$-(\nabla + i\eta_0) \cdot A(\nabla + i\eta_0)v_m(\eta_0) - (\nabla + i\eta_0) \cdot B(\nabla + i\eta_0)u_m(\eta_0) \\ - \lambda(\eta_0)v_m(\eta_0) - a_m(\eta_0)u_m(\eta_0) = 0.$$

Finally, multiply by $u_n(\eta_0)$ and integrate over Y to conclude that

$$\int_Y B(\nabla + i\eta_0)u_m(\eta_0) \cdot (\nabla - i\eta_0)\overline{u_n(\eta_0)}\, dy = a_m(\eta_0)\delta_{mn}. \tag{2.1}$$

Equation (2.1) suggests the following construction. Let $N(\eta_0)$ denote the unperturbed eigenspace, i.e.,

$$N(\eta_0) := \ker(\mathcal{A}(\eta_0) - \lambda(\eta_0)I).$$

Given a perturbation B and a basis $F = \{f_1, f_2, \ldots, f_h\}$ for $N(\eta_0)$, we can define a selfadjoint operator G_B on $N(\eta_0)$ whose matrix in the basis F is given by

$$([G_B]_F)_{m,n} := \int_Y B(\nabla + i\eta_0)f_m \cdot (\nabla - i\eta_0)\overline{f_n}\, dy.$$

In particular, it follows from equation (2.1) that in the basis of unperturbed eigenfunctions $E = \{u_1(\eta_0), u_2(\eta_0), \ldots, u_h(\eta_0)\}$, $[G_B]_E$ is a diagonal matrix,

$$[G_B]_E = diag(a_1(\eta_0), a_2(\eta_0), \ldots, a_h(\eta_0)).$$

If $[G_B]_E$ is a scalar matrix, then the operator G_B is a scalar multiple of identity operator. However, if we can find a basis F for the eigenspace and a matrix B, corresponding to which, the matrix $[G_B]_F$ has a non-zero off-diagonal entry, then for that choice of B, $[G_B]_E$ will not be a scalar matrix, and hence, $a_m(\eta_0) \neq a_n(\eta_0)$ for some $m, n \in \{1, 2, \ldots, h\}$.

Proposition 2.12. *There exists a symmetric matrix* B *with* $L^\infty_\sharp(Y,\mathbb{R})$*-entries such that the operator* G_B *is not a scalar multiple of identity.*

Proof. As mentioned earlier, the proposition will be proved if we can find a basis F and a matrix B with $L^\infty_\sharp(Y,\mathbb{R})$ entries, such that the matrix $[G_B]_F$ has a non-zero off-diagonal entry. Let $F = \{f_1, f_2, \ldots, f_h\}$ be any basis of $N(\eta_0)$, where $\eta_0 = (\eta_{0,1}, \eta_{0,2}, \ldots, \eta_{0,d})$. Observe that at least one of the following must be true: There is some $j \in \{1, 2, \ldots, d\}$, such that

$$(\partial_j + i\eta_{0,j})f_1(\partial_j - i\eta_{0,j})\overline{f_2} \not\equiv 0, \text{ or} \tag{2.2}$$

there exists $l \in \{1, 2, \ldots, d\}$, such that

$$|(\partial_l + i\eta_{0,l})f_1|^2 - |(\partial_l + i\eta_{0,l})f_2|^2 \not\equiv 0. \tag{2.3}$$

To prove this, we note that if (2.2) and (2.3) do not hold, then f_1 and f_2 are both a scalar multiple of $\exp(-i\eta_0 \cdot y)$, which contradicts the fact that they are distinct elements of a basis of $N(\eta_0)$. In either of these cases, we will choose a suitable F and B.

Case 1. If (2.2) holds, then the function g defined by $g := (\partial_j + i\eta_{0,j})f_1(\partial_j - i\eta_{0,j})\overline{f_2} \in L^1_\sharp(Y)$ because $f_1, f_2 \in H^1_\sharp(Y)$. Hence, by Hahn-Banach Theorem, there is a continuous linear functional $\kappa \in (L^1_\sharp(Y))^*$, such that $\kappa(g) = ||g||_{L^1_\sharp(Y)} \neq 0$. However, by duality, there exists a $\beta \in L^\infty_\sharp(Y)$, such that $\kappa(g) = \int_Y \beta\, g\, dy = ||g||_{L^1_\sharp(Y)} \neq 0$.

Now, either $\int_Y \mathrm{Re}(\beta) g\, dy \neq 0$ or $\int_Y \mathrm{Im}(\beta) g\, dy \neq 0$. Suppose, without loss of generality, that $\int_Y \mathrm{Re}(\beta) g\, dy \neq 0$ and define

$$B = diag(0, 0, \ldots, 0, \mathrm{Re}(\beta), 0, \ldots, 0)$$

with $\mathrm{Re}(\beta)$ in the j^{th} place, then

$$\begin{aligned}([G_B]_F)_{1,2} &= \int_Y B(\nabla + i\eta_0)f_1 \cdot (\nabla - i\eta_0)\overline{f_2}\, dy \\ &= \int_Y \mathrm{Re}(\beta)(\partial_j + i\eta_{0,j})f_1(\partial_j - i\eta_{0,j})\overline{f_2}\, dy \\ &= \int_Y \mathrm{Re}(\beta) g\, dy \neq 0.\end{aligned}$$

Case 2. If (2.3) holds, then the function g' defined by $g' := |(\partial_l + i\eta_{0,l})f_1|^2 - |(\partial_l + i\eta_{0,l})f_2|^2 \in L^1_\sharp(Y,\mathbb{R})$ because $f_1, f_2 \in H^1_\sharp(Y)$. Hence, by Hahn-Banach Theorem, there is a continuous linear functional $\kappa' \in (L^1_\sharp(Y,\mathbb{R}))^*$, such that $\kappa'(g') = ||g'||_{L^1_\sharp(Y)} \neq 0$. However, by duality, there exists a $\beta' \in L^\infty_\sharp(Y,\mathbb{R})$, such that $\kappa'(g') = \int_Y \beta' g'\, dy = ||g'||_{L^1_\sharp(Y)} \neq 0$.

Define

$$B = \mathrm{diag}(0, 0, \ldots, 0, \beta', 0, \ldots, 0)$$

with β' in the l^{th} place, then in the new basis $F' = \{f_1 + f_2, f_1 - f_2, f_3, \ldots, f_h\}$ of $N(\eta_0)$, we have

$$\begin{aligned}([G_B]_F)_{1,2} &= \int_Y B(\nabla + i\eta_0)(f_1 + f_2) \cdot (\nabla - i\eta_0)(\overline{f_1} - \overline{f_2})\ dy \\ &= \int_Y \beta' \left(|(\partial_l + i\eta_{0,l})f_1|^2 - |(\partial_l + i\eta_{0,l})f_2|^2\right)\ dy \neq 0.\end{aligned}$$

In conclusion, we have found a basis in which an off-diagonal entry of $[G_B]_F$ is non-zero. Hence, the operator G_B is not a scalar multiple of identity. In particular, the matrix $[G_B]_E$ cannot be a scalar matrix. □

Proof of Lemma 2.8. Let $A \in P_n$. Given $\epsilon > 0$, we want to find $A' \in P_{n+1}$ such that $||A - A'||_{L^\infty} < \epsilon$. We shall construct A' in the form $A' = A + \tau B$, where B is a symmetric matrix with $L^\infty_\sharp(Y, \mathbb{R})$-entries and $\tau \in \mathbb{R}$. By Lemma 2.7, we can choose τ_0 so that $A + \tau B \in P_n$ for $|\tau| < \tau_0$. Hence, the first n eigenvalues of the operator $-(\nabla + i\eta_0) \cdot (A + \tau B)(\nabla + i\eta_0)$ are simple for $|\tau| < \tau_0$. Subsequently, we must choose τ such that $|\tau| < \sigma_0 = \frac{\alpha}{2||B||_{L^\infty}}$, in order to apply the Kato-Rellich Theorem. Now, suppose that the $(n+1)^{th}$ eigenvalue of $\mathcal{A}(\eta_0)$ has multiplicity h. By Kato-Rellich Theorem (Theorem 2.11), the h eigenvalue branches of the perturbed operator $\mathcal{A}(\eta_0) + \tau\mathcal{B}(\eta_0)$ are given by the following expansion for $r = 1, 2, \ldots, h$:

$$\lambda_r(\tau; \eta_0) = \lambda(\eta_0) + \tau a_r(\eta_0) + \tau^2 \beta_r(\tau; \eta_0),$$

where $\beta_r(\tau; \eta_0)$ is real-analytic for $\tau \in (-\sigma_0, \sigma_0)$. If there are $m, n \in \{1, 2, \ldots, h\}$ such that $a_m(\eta_0) \neq a_n(\eta_0)$, then there is a τ_1 such that $\lambda_m(\tau; \eta_0) \neq \lambda_n(\tau; \eta_0)$ for $|\tau| < \tau_1$. Since two of the h eigenvalue branches are distinct for small τ, the multiplicity of the perturbed eigenvalue, which can only go down for small τ, must be less than or equal to $h - 1$. This can be achieved through an application of Proposition 2.12 which gives us a matrix B_1 such that at least two of $(a_r(\eta_0))_{r=1}^h$ are distinct. Now, starting from the matrix $A + \tau_1 B_1$, we repeat the procedure above so that the multiplicity of the $(n+1)^{th}$ eigenvalue is further reduced. The perturbed matrix is now labelled $A + \tau_1 B_1 + \tau_2 B_2$. Finally, after a finite number of such steps, we can reduce the multiplicity of the $(n+1)^{th}$ eigenvalue to 1. At the end of this procedure, we obtain a matrix of the form $A' = A + \sum_{j=1}^N \tau_j B_j$, for some $N \in \mathbb{N}$. Each perturbation must be chosen so that $\sum_{j=1}^N \tau_j ||B_j||_{L^\infty} < \epsilon$. □

Remark 2.13. Theorem 2.1 proves that an eigenvalue $\lambda(\eta_0)$ of the shifted operator $\mathcal{A}(\eta_0)$ can be made simple by a perturbation of the matrix $A \in M_B^>$. However, since the Bloch eigenvalues are Lipschitz continuous functions of the parameter $\eta \in Y'$ [CV97], the perturbed eigenvalue $\tilde{\lambda}(\eta)$ will continue to remain simple in some neighborhood of η_0.

Remark 2.14.

1. The perturbation formula (2.1) may be thought of as a variation of the Hellmann-Feynman theorem in the physics literature. The coefficients of the differential operator (1.1) are real-valued functions, in as much as they are related to properties of materials. The presence of complex-valued coefficients in the perturbation formula complicates the choice of the real-valued perturbation B.

2. In the theory of homogenization, the coefficients of the second order divergence-type periodic elliptic operator are usually only measurable and bounded. By regularity theory [LU68], the eigenfunctions of the shifted operator $\mathcal{A}(\eta)$ are known to be Hölder continuous. However, derivatives of eigenfunctions, which may not be bounded, appear in the perturbation formula (2.1). Therefore, the perturbation B is chosen using the Hahn-Banach Theorem.

2.4 Global Simplicity

In the previous section, we have proved that a given Bloch eigenvalue $\lambda_m(\eta)$ of the operator $\mathcal{A}$ can be made simple locally in Y' through a small perturbation in the coefficients. In this section, we shall perform perturbation on the operator $\mathcal{A}$ in such a way that its spectrum still retains the fibered character, i.e., $\sigma(\tilde{A}) = \cup_{\eta\in Y'}\sigma(\tilde{A}(\eta))$ and the m^{th} eigenvalue function $\eta \mapsto \tilde{\lambda}_m(\eta)$ is simple for all $\eta \in Y'$. However, the perturbed operator $\tilde{\mathcal{A}}$ may no longer be a differential operator.

Proof of Theorem 2.3. The operator (1.1) has a direct integral decomposition for periodic A, i.e., $\mathcal{A}$ is unitarily equivalent to $\int_{\eta\in\mathbb{T}^d}^{\oplus}\mathcal{A}(\eta)d\eta$, through the Gelfand transform, where $\mathcal{A}(\eta) = -(\nabla + i\eta)\cdot A(\nabla + i\eta)$ is an unbounded operator in $L^2_\sharp(Y)$. We would like to point out that Y' is understood to parametrize the torus, $\mathbb{T}^d$. Consider the m^{th} Bloch eigenvalue $\lambda_m(\eta)$ of $\mathcal{A}$. By Lemma 2.8, at any point $\eta_0 \in Y'$, we can find a perturbation of the coefficients $A = (a_{kl})$ of $\mathcal{A}(\eta_0)$ so that the perturbed eigenvalue $\tilde{\lambda}_m(\eta_0)$ is simple. By Remark 2.13, there is a neighborhood $\mathcal{G}_{\eta_0}$ of η_0, in which the perturbed eigenvalue $\tilde{\lambda}_m(\eta)$ of the perturbed shifted operator $\tilde{\mathcal{A}}(\eta)$ is simple. In this manner, for each $\xi \in \mathbb{T}^d$, we obtain a perturbation B_ξ and a neighborhood $\mathcal{G}_\xi$ in which the eigenvalue of the perturbed operator $\tilde{\mathcal{A}}(\eta) = -(\nabla + i\eta) \cdot (A + B_\xi)(\nabla + i\eta)$ is simple. The collection $\{G_\xi\}_{\xi \in \mathbb{T}^d}$ is an open covering of the torus. By compactness of $\mathbb{T}^d$, there is a finite subcovering having the property that in each member $\mathcal{G}_\xi$ of the subcovering, the corresponding perturbation B_ξ causes the perturbed eigenvalue $\tilde{\lambda}_m(\eta)$ to be simple in $\mathcal{G}_\xi$.

Let $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$ be the finite subcovering of the torus obtained above. Define $\mathcal{O}_1 = \mathcal{G}_1$. For $r \geq 1$, define $\mathcal{O}_{r+1} = \mathcal{G}_{r+1} \setminus \bigcup_{j=1}^{r} \mathcal{G}_j$. Suppose that B_j is the perturbation corresponding to the set $\mathcal{O}_j$.

For every $j \in \{1, 2, \ldots, n\}$, let $\chi_{\mathcal{O}_j}$ denote the characteristic function of the set $\mathcal{O}_j$. Now, define the parametrized operator

$$\tilde{\mathcal{A}}(\eta) = -(\nabla + i\eta) \cdot (A + \sum_{j=1}^{n} B_j \chi_{\mathcal{O}_j})(\nabla + i\eta)$$

which depends measurably on $\eta \in \mathbb{T}^d$. Finally, define the direct integral $\tilde{\mathcal{A}} = \int_{\eta \in \mathbb{T}^d}^{\oplus} \tilde{\mathcal{A}}(\eta) d\eta$, where each of the fibers is a differential operator in $L^2_\sharp(Y)$. Then it is known [RS78, p.284] that

$$\sigma(\tilde{A}) = \bigcup_{\eta \in Y'} \sigma(\tilde{A}(\eta)).$$

Hence, we may define an m^{th} eigenvalue function $\eta \mapsto \tilde{\lambda}_m(\eta)$ with the property that

$$|\lambda_m(\eta) - \tilde{\lambda}_m(\eta)| \leq C \max_{1 \leq j \leq n} ||B_j||_{L^\infty},$$

where $\lambda_m(\eta)$ is the m^{th} Bloch eigenvalue of $\mathcal{A}$. □

Remark 2.15.

1. Although the m^{th} eigenvalue of the perturbed operator is simple for all parameter values, $\tilde{\lambda}_m(\eta)$ may only be measurable in $\eta \in Y'$. However, $\tilde{\lambda}_m(\eta)$ is analytic in each $\mathcal{O}_j \subset \mathbb{T}^d$.

2. The perturbed operator $\tilde{\mathcal{A}}$ is no longer a differential operator, even though each fiber $\tilde{\mathcal{A}}(\eta)$ is a differential operator. In fact, we shall prove in Theorem 2.17 that $\tilde{\mathcal{A}}$ is a differential operator if and only if $B_1 = B_2 = \ldots = B_n$.

3. A rigorous account of direct integral decomposition of operators, such as the one employed above for periodic operators, may be found in [Sch90] and [Mau68].

Lemma 2.16. *Let* B *be a symmetric matrix with* $L^\infty_\sharp(Y,\mathbb{R})$*-entries. Define* $\mathcal{B}(\eta) = -(\nabla + i\eta)\cdot B(\nabla + i\eta)$*. Let* $O \subset Y'$ *be a proper subset of* Y' *such that* B *is not identically zero on* O*. Then the operator* $\mathcal{B}$*, defined through its unitary equivalence to the direct integral* $\int^{\bigoplus}_{\eta\in\mathbb{T}^d} \mathcal{B}(\eta)\chi_O \, d\eta$*, is not a differential operator.*

Proof. By Peetre's Theorem [Pee60], [Nar85, p. 174], a linear operator $\mathcal{B} : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ is locally a differential operator if and only if $supp(\mathcal{B}u) \subset supp(u)$ for all $u \in \mathcal{D}(\mathbb{R}^d)$. Here, $\mathcal{D}(\mathbb{R}^d)$ denotes the space of compactly supported smooth functions on $\mathbb{R}^d$ with the topology of test functions. Also, let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^d$. In order to show that $\mathcal{B}$ is not a differential operator, we will show that it does not preserve supports.

Given $g \in \mathcal{D}(\mathbb{R}^d)$, we define its Gelfand transform as

$$g_\sharp(y,\eta) = \sum_{p\in\mathbb{Z}^d} g(y + 2\pi p)e^{-i(y+2\pi p)\cdot\eta}.$$

This is a function in $L^2(Y', L^2_\sharp(Y))$. The map $g \mapsto g_\sharp$ is an isometry from $\mathcal{D}(\mathbb{R}^d)$, equipped with the L^2-norm, to $L^2(Y', L^2_\sharp(Y))$ and hence it may be extended to a unitary isomorphism from $L^2(\mathbb{R}^d)$ to $L^2(Y', L^2_\sharp(Y))$. We shall show that $\mathcal{B}(g)$ is not compactly supported. Note that $\mathcal{B}(g)$ is a tempered distribution and it may be defined as:

$$(\mathcal{B}(g), \phi) = \int_O \int_Y B(\nabla + i\eta)g_\sharp(y,\eta) \cdot (\nabla - i\eta)\overline{\phi}_\sharp(y,\eta) \; dy\, d\eta.$$

We may define the Fourier transform of $\mathcal{B}(g)$ in $\mathcal{S}'(\mathbb{R}^d)$ as

$$(\widehat{\mathcal{B}(g)}, \phi) = (\mathcal{B}(g), \mathcal{F}^{-1}(\phi)),$$

where $\mathcal{F}^{-1}(\phi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(\eta)e^{iy\cdot\eta} \, d\eta$ is the inverse Fourier transform of ϕ. Since $\phi \in \mathcal{S}(\mathbb{R}^d)$, there exists a $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\phi = \widehat{\psi}$. Therefore,

$$(\widehat{\mathcal{B}(g)}, \phi) = (\mathcal{B}(g), \mathcal{F}^{-1}(\phi)) = (\mathcal{B}(g), \psi).$$

By Poisson Summation Formula [Gra08, p. 171], we conclude that

$$\begin{aligned} \psi_\sharp(y,\eta) = \sum_{p\in\mathbb{Z}^d} \psi(y + 2\pi p)e^{-i(y+2\pi p)\cdot\eta} &= \frac{1}{(2\pi)^{d/2}} \sum_{q\in\mathbb{Z}^d} \widehat{\psi}(\eta + q)e^{iq\cdot y} \\ &= \frac{1}{(2\pi)^{d/2}} \sum_{q\in\mathbb{Z}^d} \phi(\eta + q)e^{iq\cdot y}. \end{aligned} \tag{2.4}$$

Now, suppose that $\phi \in \mathcal{S}(\mathbb{R}^d)$ vanishes on $\bigcup_{q\in\mathbb{Z}^d}(\mathcal{O}+q)$, then $\psi_\sharp$, as obtained in (2.4), vanishes on $\mathcal{O}$. Hence,

$$(\widehat{\mathcal{B}(g)},\phi) = (\mathcal{B}(g),\psi) = \int_Y\int_{\mathcal{O}} B(\nabla+i\eta)g_\sharp(y,\eta)\cdot(\nabla-i\eta)\overline{\psi}_\sharp(y,\eta)\ d\eta\, dy = 0.$$

Therefore, $\widehat{\mathcal{B}(g)}$ vanishes on the open set $\mathbb{R}^d\setminus\bigcup_{q\in\mathbb{Z}^d}(\overline{\mathcal{O}}+q)$. By Schwartz-Paley-Wiener Theorem [Rud91, p. 191], $\widehat{\mathcal{B}(g)}$ cannot be the Fourier transform of a compactly supported distribution, i.e., $\mathcal{B}(g)$ is not compactly supported. □

Theorem 2.17. *Let* $\{\mathcal{O}_1,\mathcal{O}_2,\ldots,\mathcal{O}_n\}$ *be a partition of* Y' *up to a set of measure zero, i.e.,* $Y'\setminus\bigcup_{j=1}^n\mathcal{O}_j$ *is a set of measure zero. Define* $\mathcal{B}:\mathcal{D}(\mathbb{R}^d)\to\mathcal{D}'(\mathbb{R}^d)$ *by* $\mathcal{B}(g)=\sum_{j=1}^n\int_{\eta\in\mathcal{O}_j}^{\oplus}\mathcal{B}_j(\eta)g_\sharp(y,\eta)\,d\eta$ *where* $\mathcal{B}_j(\eta)=-(\nabla+i\eta)\cdot B_j(\nabla+i\eta)$ *for all* $j\in\{1,2,\ldots,n\}$, B_j *are matrices with* $L^\infty_\sharp(Y,\mathbb{R})$*-entries, then* $\mathcal{B}$ *is a differential operator if and only if* $B_1=B_2=\ldots=B_n$.

Proof. If $B:=B_1=B_2=\ldots=B_n$, then $\mathcal{B}(g)=-\nabla\cdot B\nabla(g)$ which is a differential operator.

Conversely, without loss of generality, assume that $B_1\neq B_2$ and suppose that $\mathcal{B}$ is a differential operator. Then

$$\mathcal{B}(g)=\int_{\eta\in Y'}^{\oplus}\mathcal{B}_1(\eta)d\eta+\int_{\eta\in\mathcal{O}_2}^{\oplus}(\mathcal{B}_2-\mathcal{B}_1)(\eta)d\eta+\int_{\eta\in\mathcal{O}_3}^{\oplus}(\mathcal{B}_3-\mathcal{B}_1)(\eta)d\eta+\ldots$$
$$+\int_{\eta\in\mathcal{O}_n}^{\oplus}(\mathcal{B}_n-\mathcal{B}_1)(\eta)d\eta.$$

Hence,

$$\mathcal{B}(g)-\int_{\eta\in Y'}^{\oplus}\mathcal{B}_1(\eta)d\eta=\sum_{j=2}^{n}\int_{\eta\in\mathcal{O}_j}^{\oplus}(\mathcal{B}_j-\mathcal{B}_1)(\eta)d\eta$$

The left hand side of the above equation is a differential operator. We will show that the right hand side is not a differential operator to obtain a contradiction.

We proceed as in Lemma 2.16.

Define $\mathcal{C}:\mathcal{D}(\mathbb{R}^d)\to\mathcal{D}'(\mathbb{R}^d)$ by

$$(\mathcal{C}(g),\phi)=\sum_{j=2}^{n}\int_{\mathcal{O}_j}\int_Y(B_j-B_1)(\nabla+i\eta)g_\sharp(y,\eta)\cdot(\nabla-i\eta)\overline{\phi}_\sharp(y,\eta)\ dy\,d\eta$$

It is easy to see that $\mathcal{C}(g)\in\mathcal{S}'(\mathbb{R}^d)$.

Therefore, we may define its Fourier transform by

$$(\widehat{\mathcal{C}(g)},\phi)=(\mathcal{C}(g),\mathcal{F}^{-1}(\phi)),$$

where $\mathcal{F}^{-1}(\phi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(\eta) e^{iy\cdot\eta}\, d\eta$ is the inverse Fourier transform of ϕ. Since $\phi \in \mathcal{S}(\mathbb{R}^d)$, there exists $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\phi = \widehat{\psi}$. Therefore,

$$(\widehat{C(g)}, \phi) = (C(g), \mathcal{F}^{-1}(\phi)) = (C(g), \psi).$$

As in (2.4), we have

$$\psi_\sharp(y,\eta) = \frac{1}{(2\pi)^{d/2}} \sum_{q\in\mathbb{Z}^d} \phi(\eta + q) e^{iq\cdot y}. \tag{2.5}$$

Now, suppose that $\phi \in \mathcal{S}(\mathbb{R}^d)$ vanishes on $\bigcup_{q\in\mathbb{Z}^d}(\bigcup_{j=2}^n \mathcal{O}_j + q)$, then $\psi_\sharp$, as obtained in (2.5), vanishes on $\bigcup_{j=2}^n \mathcal{O}_j$. Hence,

$$\begin{aligned}(\widehat{C(g)}, \phi) &= (C(g), \psi)\\ &= \sum_{j=2}^n \int_Y \int_{\mathcal{O}_j} (B_j - B_1)(\nabla + i\eta) g_\sharp(y,\eta) \cdot \overline{(\nabla - i\eta)\psi_\sharp}(y,\eta)\, d\eta\, dy = 0.\end{aligned}$$

Therefore, $\widehat{C(g)}$ vanishes on the open set $\mathbb{R}^d \setminus \bigcup_{q\in\mathbb{Z}^d}(\bigcup_{j=2}^n \overline{\mathcal{O}}_j + q)$. By Schwartz-Paley-Wiener Theorem [Rud91, p. 191], $\widehat{C(g)}$ cannot be the Fourier transform of a compactly supported distribution, i.e., $C(g)$ is not compactly supported. Therefore, C is not a differential operator. □

2.5 Proof of Theorem 2.4

In this section, we prove that a spectral edge of a periodic elliptic differential operator can be made simple through a perturbation in the coefficients. The proof essentially follows Klopp and Ralston [KR00], with the straightforward modification that the coefficients must come from $W^{1,\infty}_\sharp(Y, \mathbb{R})$. This condition is required to ensure that the eigenfunctions and their derivatives are Hölder continuous functions. We produce the proof here for completeness.

Suppose that the coefficients of the operator (1.1), $a_{kl} \in W^{1,\infty}_\sharp(Y)$. Note that the Bloch eigenvalues which are defined for $\eta \in Y'$ are Lipschitz continuous in η and may be extended as periodic functions to $\mathbb{R}^d$. In the sequel, we shall treat the Bloch eigenvalues as functions on $\mathbb{T}^d$, which is identified with Y' in a standard way. Also, we shall write $\lambda_j(\eta, A)$ to specify that a Bloch eigenvalue corresponds to a particular matrix A, appearing in the operator $\mathcal{A}$. We shall prove the theorem for an upper endpoint of a spectral gap. The proof for a lower endpoint is identical. We shall require the following lemma.

Lemma 2.18. *Consider the operator* $\mathcal{A}$ *as in* (1.1)*, with* $A \in M_B^>$. *Let* λ_0 *correspond to the upper edge of a spectral gap of* $\mathcal{A}$ *and let* m *be the smallest index such that the Bloch eigenvalue* λ_m *attains* λ_0*, then*

(L1) There exist numbers $a, b \in \mathbb{R}$ *such that* $\lambda_{m-1}(\eta) < a < \lambda_0 \leq \lambda_m(\eta) < b$ *for all* $\eta \in Y'$. *Further, there exists* $M \in \mathbb{N}$ *such that* $M > m$ *and the Bloch eigenvalue* λ_M *satisfies* $\lambda_M(\eta) > b$ *for all* $\eta \in Y'$.

(L2) Let B *be a symmetric matrix with* $L^\infty_\sharp(Y, \mathbb{R})$*-entries. There is a finite open covering of* Y', $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$ *such that for each* $\mathcal{G}_j$*, we have an orthonormal set in* $L^2_\sharp(Y)$ *of functions real-analytic for* $\eta \in \mathcal{G}_j$ *and for sufficiently small* t,

$$\{\phi_m^{(j)}(\eta, A - tB), \phi_{m+1}^{(j)}(\eta, A - tB), \ldots, \phi_{R_j}^{(j)}(\eta, A - tB)\}. \tag{2.6}$$

Further, for each fixed t*, the linear subspace generated by the functions in* (2.6) *contains the eigenspaces corresponding to eigenvalues of* $-(\nabla + i\eta) \cdot (A - tB)(\nabla + i\eta)$ *between* a *and* b.

(L3) The functions in (2.6) *may be chosen such that the following equation is satisfied*

$$\left\langle \frac{d\phi_r^{(j)}}{dt}, \phi_s^{(j)} \right\rangle = 0 \qquad \text{for} \qquad r, s \in \{m, m+1, \ldots, R_j\}, \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ *denotes the* $L^2_\sharp(Y)$ *inner product.*

Proof.

Proof of (L1) As noted in Remark 2.13, the Bloch eigenvalues are Lipschitz continuous functions on a compact set $\mathbb{T}^d$. Hence, the function $\eta \mapsto \lambda_m(\eta)$ is bounded above, say by b. Since λ_0 is a spectral edge, $\delta := \min_{\eta \in Y'} \lambda_m(\eta) - \max_{\eta \in Y'} \lambda_{m-1}(\eta)$ is positive. Choose $a = \lambda_0 - \frac{\delta}{2}$. These choices of a and b satisfy our requirements.

By application of Weyl's law [RS78], the eigenvalues of the periodic Laplacian on Y satisfy the following inequality, for some $s > 0$ and $C_1 > 0$, for large M,

$$\lambda_M(0, I) \geq \lambda_M^N \geq C_1 M^s, \tag{2.8}$$

where λ_M^N denotes the M^{th} eigenvalue of the Neumann Laplacian on Y and I is the d-dimensional identity matrix. Note that $\lambda_M(\eta, I)$ denotes the M^{th} Bloch eigenvalue of the periodic Laplacian.

By Lipschitz continuity of Bloch eigenvalues in the dual parameter, we have

$$|\lambda_M(\eta, I) - \lambda_M(0, I)| \leq C|\eta| \leq C_2.$$

Therefore, for all $\eta \in Y'$,

$$\lambda_M(\eta, I) \geq \lambda_M(0, I) - C_2. \tag{2.9}$$

On combining (2.8) and (2.9), for all $\eta \in Y'$, we obtain

$$\lambda_M(\eta, I) \geq C_1 M^s - C_2.$$

It follows from a standard argument involving min-max principle, that $\lambda_M(\eta, I) \leq ||A^{-1}||_{L^\infty} \lambda_M(\eta, A)$.

Therefore, for all $\eta \in Y'$,

$$\begin{aligned} \lambda_M(\eta, A) &\geq \frac{1}{||A^{-1}||_{L^\infty}} \lambda_M(\eta, I) \\ &\geq \frac{C_1 M^s}{||A^{-1}||_{L^\infty}} - \frac{C_2}{||A^{-1}||_{L^\infty}}. \end{aligned}$$

Finally to prove (L1), choose M large enough so that

$$\frac{C_1 M^s}{||A^{-1}||_{L^\infty}} - \frac{C_2}{||A^{-1}||_{L^\infty}} > b.$$

Proof of (L2) In order to prove (L2), we shall construct a set of functions, real-analytic in a neighborhood of each point of the torus and for small t, spanning the union of eigenspaces corresponding to eigenvalues of the operator $\mathcal{A}(\eta; A - tB) := -(\nabla + i\eta) \cdot (A - tB)(\nabla + i\eta)$ between a and b and then employ compactness of $\mathbb{T}^d$ to obtain a finite collection of neighborhoods.

For each $\xi \in \mathbb{T}^d$, there is a circle Γ_ξ in the complex plane containing the eigenvalues of $\mathcal{A}(\xi, A)$ between a and b. Let B be a $d \times d$ real symmetric matrix with $W^{1,\infty}_\sharp(Y)$ entries. There is a neighborhood R_ξ of ξ such that the number of eigenvalues of operator $\mathcal{A}(\eta; A - tB)$ between a and b remains constant for $\eta \in R_\xi$ and for small t. As a consequence, the operator P_ξ defined by

$$P_\xi(\eta; A - tB) := -\frac{1}{2\pi i} \int_{\Gamma_\xi} \left(\mathcal{A}(\eta; A - tB) - zI\right)^{-1} \, dz \tag{2.10}$$

is real-analytic in R_ξ of ξ and for small t. The operator P_ξ is an orthogonal projection onto the direct sum of eigenspaces of $\mathcal{A}(\eta; A - tB)$ corresponding to the eigenvalues between a and b. The analyticity of the projection operator follows from the analyticity of the integrand, which is a consequence of the operator family $\mathcal{A}(\eta; A - tB)$ being a holomorphic family of type (B).

In order to obtain a collection of functions, real-analytic for $\eta \in R_\xi$ and small t, we choose an orthonormal set of eigenfunctions of $\mathcal{A}(\xi, A)$ corresponding to eigenvalues between a and b, for example, let an element of this set be $\phi(\cdot, \xi, 0)$, then the function defined by $\phi(\cdot, \eta, t) := P_\xi(\eta; A - tB)\phi(\cdot, \xi, 0)$ belongs to the linear subspace containing eigenfunctions of $\mathcal{A}(\eta; A - tB)$ corresponding to eigenvalues between a and b in R_ξ for small t and is analytic as a $L^2_\sharp(Y)$-valued map. There are two technical issues.

Firstly, it remains to prove that this mapping is also analytic when viewed as $H^1_\sharp(Y)$-valued map. For the proof, we require the following facts. (i) The map $(\eta, t) \to \|\phi(\cdot, \eta, t)\|_{H^1_\sharp(Y)}$ is bounded for $\eta \in R_\xi$ and small t. Note that $\phi(\cdot, \eta, t)$ is a linear combination of eigenfunctions corresponding to eigenvalues between a and b. Therefore, $\|\phi(\cdot, \eta, t)\|_{H^1_\sharp(Y)}$ is bounded due to boundedness of those eigenfunctions in $H^1_\sharp(Y)$. (ii) $L^2_\sharp(Y)$ is dense in $H^{-1}_\sharp(Y)$. The two facts imply analyticity of the map from η to $H^1_\sharp(Y)$. Indeed, weak-holomorphy for a Banach space-valued map can be proved using weak-holomorphy for a dense subspace, provided boundedness of the map. More details can be found in [SGV04, p.23, Lemma 2.5].

Secondly, the application of the projection operator may lead to loss of normality. However, we can further impose norm 1 condition by dividing by the norm, which is an analytic function away from 0.

In this manner, for each $\xi \in \mathbb{T}^d$, we have a neighborhood R_ξ with certain properties. These sets form an open covering of $\mathbb{T}^d$. By compactness of $\mathbb{T}^d$, the open covering has a finite subcovering $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n\}$ with the following properties.

1. For each $\mathcal{G}_j$, we have an orthonormal set in $L^2_\sharp(Y)$

   $$\{\phi^{(j)}_m(\eta, A - tB), \dots, \phi^{(j)}_{R_j}(\eta, A - tB)\}$$

   whose elements are real-analytic for $\eta \in \mathcal{G}_j$ and small t.

2. The linear subspace generated by

   $$\{\phi^{(j)}_m(\eta, A - tB), \dots, \phi^{(j)}_{R_j}(\eta, A - tB)\}$$

   contains the eigenspaces corresponding to eigenvalues of $\mathcal{A}(\eta; A - tB)$ that lie between a and b.

Proof of (L3) Let $\tilde{\phi}^{(j)}_r = \sum_{p=m} u^{(j)}_{rp}\phi^{(j)}_p$, then

$$\left\langle \frac{d\tilde{\phi}^{(j)}_r}{dt}, \tilde{\phi}^{(j)}_s \right\rangle = \sum_p \frac{du^{(j)}_{rp}}{dt}\overline{u^{(j)}_{sp}} + \sum_{p,q} u^{(j)}_{rp}\overline{u^{(j)}_{sq}} \left\langle \frac{d\phi^{(j)}_p}{dt}, \phi^{(j)}_q \right\rangle,$$

where $u^{(j)}_{rp} = u^{(j)}_{rp}(t)$ are functions of t.

If we set U to be the matrix with entries $u_{rs}^{(j)}$ and Ψ to be the matrix with entries $-\left\langle \frac{d\phi_r^{(j)}}{dt}, \phi_s^{(j)} \right\rangle$, (2.7) will hold if

$$\frac{dU}{dt} = U\Psi.$$

This is solved with the initial condition $U(0) = I$, where I is the identity matrix of dimension $R_j - m + 1$. The matrix Ψ is antiselfadjoint, therefore, $U(t)$ is unitary and analytic for $\eta \in \mathcal{G}$. Replace $\phi_r^{(j)}$ with $\tilde{\phi}_r^{(j)}$ to complete the proof of (L3). □

Proof of Theorem 2.4.

Consider the sesquilinear form

$$a(\eta, t)(u, v) := \int_Y (A - tB)(\nabla + i\eta)u \cdot (\nabla - i\eta)\bar{v}.$$

For the functions constructed in (L2), $\langle \frac{d\phi_r^{(j)}}{dt}, \phi_s^{(j)} \rangle = 0$ for all r, s. Thus,

$$\begin{aligned} &\frac{d}{dt}\left(a(\eta, t)(\phi_r^{(j)}(\eta, A - tB), \phi_s^{(j)}(\eta, A - tB))\right) \\ &\quad = -\int_Y B(\nabla + i\eta)\phi_r^{(j)}(\eta, A - tB) \cdot (\nabla - i\eta)\overline{\phi_s^{(j)}}(\eta, A - tB)\, dy. \end{aligned}$$

A function f defined on $\mathbb{R}^d$ is said to be (η, Y)-periodic if for all $p \in \mathbb{Z}^d$, $y \in \mathbb{R}^d$, $u(y + 2\pi p) = e^{2\pi i p\cdot\eta}u(y)$. The eigenfunctions of $\mathcal{A}(\eta, A)$ with periodic boundary conditions, when multiplied by $\exp(-i\eta \cdot y)$, become eigenfunctions of $\mathcal{A} := -\nabla \cdot A\nabla$ with (η, Y)-periodic boundary conditions, i.e., there are λ and u such that $-\nabla \cdot (A\nabla)u = \lambda u$, where u is (η, Y)-periodic. Since u is a complex-valued function, the regularity theorem [LU68, Chapter 3, Section 15] cannot be applied directly. However, since the operator is linear, we may write $u = v + iw$ and express the eigenvalue equation for u as two equations for the real-valued functions v and w. In particular, v and w satisfy $-\nabla \cdot (A\nabla)v = \lambda v$ and $-\nabla \cdot (A\nabla)w = \lambda w$ in the interior of Y. Hence, by the regularity theory for elliptic equations with $W^{1,\infty}$ coefficients, v and w and their first-order derivatives are Hölder continuous in the interior of Y. Further, the Hölder estimates in the interior of Y are independent of $\eta \in Y'$. Consequently, u and its derivatives are Hölder continuous in the interior of Y.

Choose $\widehat{\eta}$ and ϕ_0 such that $\mathcal{A}(\widehat{\eta}, A)\phi_0 = \lambda_0\phi_0$. Choose $\phi_0 \neq \exp(-i\widehat{\eta}\cdot y)$. This can be achieved because the multiplicity of the Bloch eigenvalue at $\widehat{\eta}$ is greater than one. Therefore, $(\nabla + i\widehat{\eta})\phi_0$ is non-zero. Consequently, there exist a y_0 in the interior of Y, an l with $1 \le l \le d$, and a $\theta > 0$ such that $\left|\left(\frac{\partial}{\partial x_l} + i\widehat{\eta}_l\right)\phi_0(y_0, \widehat{\eta})\right|^2 \geq \theta$.

Since ϕ_0 and its derivatives are Hölder continuous in the interior of Y, there is a small $\epsilon_0 > 0$ such that,

$$\text{for } |y - y_0| < \epsilon_0, \qquad \left|\left(\frac{\partial}{\partial x_l} + i\widehat{\eta}_l\right)\phi_0(y,\widehat{\eta})\right|^2 > \frac{2\theta}{3}. \tag{2.11}$$

Additionaly, since $\phi_r^{(j)}$ obtained earlier in (L2) are linear combinations of eigenfunctions, by the Hölder continuity of the eigenfunctions and their derivatives, an ϵ_0 may be chosen so that

$$\sum_{p=m}^{R_j}\left|\left(\frac{\partial}{\partial x_l} + i\eta_l\right)\phi_p^{(j)}(y,\eta,A) - \left(\frac{\partial}{\partial x_l} + i\eta_l\right)\phi_p^{(j)}(y_0,\eta,A)\right|^2 < \frac{\theta}{3}, \tag{2.12}$$

for $\eta \in \mathcal{G}_j$ and $|y - y_0| < \epsilon_0$. Define the matrix $B = diag(0,\ldots,0,b_l,0,\ldots,0)$ all of whose diagonal entries are zero other than b_l which is chosen as a function $b_l \in C_0^\infty(|y - y_0| < \epsilon_0)$ such that $b_l \geq 0$ and $\int_Y b_l = 1$. Extend B periodically to $\mathbb{R}^d$.

There is an index q such that $\widehat{\eta} \in \mathcal{G}_q$. Therefore, $\phi_0(y,\widehat{\eta}) = \sum_{r=m}^{R_q} c_r\phi_r^{(q)}(y,\widehat{\eta},A)$.

Define $\phi_0(y,\widehat{\eta},t) = \sum_{r=m}^{R_q} c_r\phi_r^{(q)}(y,\widehat{\eta},A - tB)$. Then by (2.11),

$$\begin{aligned}
&\frac{d}{dt}\left(a(\widehat{\eta},t)(\phi_0(\cdot,\widehat{\eta},t),\phi_0(\cdot,\widehat{\eta},t))\right)|_{t=0} \\
&\quad = -\int_Y b_l\left(\frac{\partial}{\partial y_l} + i\widehat{\eta}_l\right)\phi_0(y,\widehat{\eta})\left(\frac{\partial}{\partial y_l} - i\widehat{\eta}_l\right)\overline{\phi_0(y,\widehat{\eta})}\;dy \\
&\quad \leq \frac{-2\theta}{3}.
\end{aligned}$$

Hence, the following holds for t sufficiently small:

$$a(\widehat{\eta},t)(\phi_0(\cdot,\widehat{\eta},t),\phi_0(\cdot,\widehat{\eta},t)) \leq \lambda_0 - \frac{2\theta}{3}t + t^2\beta(t), \tag{2.13}$$

where $\beta(t) = O(1)$ as $t \to 0$.

Given $\eta \in Y'$, there is $j \in \{1,2,\ldots,n\}$ such that $\eta \in \mathcal{G}_j$. We define the function

$$\phi_*^{(j)}(y,\eta,t) = \sum_{r=m}^{R_j}\overline{(\partial_l + i\eta_l)\,\phi_r^{(j)}(y_0,\eta,A)}\phi_r^{(j)}(y,\eta,A - tB). \tag{2.14}$$

For $\phi(\cdot,\eta,t) = \sum_{r=m}^{R_j} a_r^{(j)}(\eta)\phi_r^{(j)}(\cdot,\eta,A - tB)$, $\phi(\cdot,\eta,t)$ is perpendicular to $\phi_*^{(j)}(\cdot,\eta,t)$ if and only if

$$\sum_{r=m}^{R_j} a_r^{(j)}(\eta)\,(\partial_l + i\eta_l)\,\phi_r^{(j)}(y_0,\eta,A) = 0. \tag{2.15}$$

Further, $a_r^{(j)}(\eta)$ may be chosen analytic for $\eta \in \mathcal{G}_j$. For $\phi(\cdot,\eta,t)$ satisfying (2.15) and $\|\phi\|_{L^2_\sharp(Y)} = 1$, the following holds for $\eta \in \mathcal{G}_j$,

$$\begin{aligned}
&\frac{d}{dt}\left(a(\eta,t)(\phi(\cdot,\eta,t),\phi(\cdot,\eta,t))\right)|_{t=0} \\
&= -\int_Y b_l\left(\frac{\partial}{\partial y_l} + i\eta_l\right)\phi(y,\eta,0)\left(\frac{\partial}{\partial y_l} - i\eta_l\right)\overline{\phi(y,\eta,0)}\, dy \\
&= -\int_{B_{\epsilon_0}(y_0)}\left|\sum_{r=m}^{R_j} a_r^{(j)}(\eta)\left((\partial_l + i\eta_l)\,\phi_r^{(j)}(y,\eta,A) - (\partial_l + i\eta_l)\,\phi_r^{(j)}(y_0,\eta,A)\right)\right|^2 b_l\, dy \\
&\geq -\frac{\theta}{3},
\end{aligned} \tag{2.16}$$

where the last inequality follows from (2.12). The right hand side of the above inequality is independent of $j \in \{1,2,\ldots,n\}$ and hence it holds for all $\eta \in Y'$. Therefore, the following holds true for sufficiently small t, uniformly for $\eta \in Y'$ and $\|\phi\|_{L^2_\sharp(Y)} = 1$,

$$a(\eta,t)(\phi(\cdot,\eta,t),\phi(\cdot,\eta,t)) \geq \lambda_0 - \frac{\theta}{3}t + t^2\gamma(t). \tag{2.17}$$

In particular, $\gamma(t)$ is bounded uniformly for $\eta \in Y'$ and small t.

To find an upper bound for $\lambda_m(\widehat{\eta}, A - tB)$, we apply the following variational characterization of the eigenvalues of $\mathcal{A}(\eta, A-tB)$ to (2.13). If $\phi_1,\phi_2,\ldots,\phi_{m-1}$ are the first $m-1$ eigenfunctions corresponding to the selfadjoint operator $\mathcal{A}(\eta, A-tB)$, then the m^{th} eigenvalue of $\mathcal{A}(\eta, A-tB)$ is given by the formula

$$\lambda_m(\eta, A - tB) = \min_{\phi\perp\{\phi_1,\phi_2,\ldots,\phi_{m-1}\},\ \|\phi\|_{L^2_\sharp(Y)}=1} a(\eta,t)(\phi,\phi).$$

Therefore, by (2.13),

$$\lambda_m(\widehat{\eta}, A - tB) < \lambda_0 - \frac{7\theta}{12}t, \tag{2.18}$$

for t sufficiently small. To find a lower bound for $\lambda_{m+1}(\eta, A-tB)$, we apply another variational characterization for the eigenvalues to (2.17), viz.,

$$\lambda_{m+1}(\eta, A - tB) = \max_{\dim V = m}\ \min_{\phi\perp V,\ \|\phi\|_{L^2_\sharp(Y)}=1} a(\eta,t)(\phi,\phi), \tag{2.19}$$

where V varies over m-dimensional subspaces of $H^1_\sharp(Y)$.

For each fixed η and t, take the m-dimensional subspace $V_{\eta,t}$ spanned by the first $m-1$ eigenfunctions of $\mathcal{A}(\eta, A-tB)$ and $\phi_*^{(j)}$ as defined in (2.14), i.e.,

$$V_{\eta,t} = \{\phi_1(\eta,t), \phi_2(\eta,t), \ldots, \phi_{m-1}(\eta,t), \phi_*^{(j)}(\eta,t)\}.$$

Then $\phi(\eta, t)$ satisfying the equation (2.15) is perpendicular to $V_{\eta,t}$ and allows us to conclude that

$$\lambda_{m+1}(\eta, A - tB) > \lambda_0 - \frac{5\theta}{12}t, \tag{2.20}$$

for sufficiently small t and for all $\eta \in Y'$. The new spectral edge is still attained by the m^{th} Bloch eigenvalue for small t and is its minimum value. Therefore, the estimate (2.18) implies that the new spectral edge is smaller than the unperturbed spectral edge. In particular, the new spectral edge $\tilde{\lambda}_0$ satisfies $\tilde{\lambda}_0 < \lambda_0 - \frac{7\theta}{12}t$ for small t. The estimate (2.20) implies that the perturbed $(m+1)^{th}$ Bloch eigenvalue lies above $\lambda_0 - \frac{5\theta}{12}t$ for small t. Hence, the distance between the new spectral edge $\tilde{\lambda}_0$ and the perturbed $(m+1)^{th}$ Bloch eigenvalue is at least $\frac{\theta}{6}t$ for small t. As a consequence, the new spectral edge is only attained by the m^{th} Bloch eigenvalue, i.e., it is simple. □

Remark 2.19. The proof of Theorem 2.4 depends crucially on the interior Hölder continuity of the Bloch eigenfunctions and their derivatives. This requires the coefficients of the elliptic operator to have $W^{1,\infty}_{\sharp}(Y)$ entries. We attempt to reduce this regularity requirement to L^∞ in Section 2.6.

2.6 Proof of Theorem 2.5

We shall prove Theorem 2.5 for an upper endpoint of a spectral gap. The proof for a lower endpoint is identical. Let λ_0 be the upper endpoint of a spectral gap of $\mathcal{A} := -\nabla \cdot (A\nabla)$, which is achieved by the Bloch eigenvalue $\lambda_m(\eta)$ at finitely many points $\eta_1, \eta_2, \dots, \eta_N$ in Y'. The proof uses ideas from Parnovski and Shterenberg [PS17] and is divided into the following steps:

1. Lemma 2.22 shows that we can find neighborhoods (uniform in t) of $\eta_1, \eta_2, \dots, \eta_N$ in Y' such that the perturbed Bloch eigenvalue is simple at all those points in them where the spectral edge is attained.

2. However, there may be points outside these neighborhoods where the spectral edge is attained. In Lemma 2.23, we show that we can choose t_0 such that for all $t \in (0, t_0]$, all points where spectral edge is attained are in a desired union of neighborhoods. In particular, we can choose these to be the neighborhoods found in Step 1.

We shall require the following preliminaries.

The multiplicity of a Bloch eigenvalue can be reduced at a finite number of points in the dual parameter by application of the same perturbation. This will be the content of the next proposition.

Proposition 2.20. *Fix* $m \in \mathbb{N}$. *Let* $S = \{\eta_1, \eta_2, \dots, \eta_N\}$ *be a finite collection of points in* Y'. *Then there exists a matrix* B *with* $L^\infty_\sharp(Y, \mathbb{R})$*-entries and a* t_0 *positive such that for all* $t \in (0, t_0]$, *the Bloch eigenvalue* $\lambda_m(\eta; A + tB)$ *of the operator* $\mathcal{A} + t\mathcal{B} = -\nabla \cdot (A + tB)\nabla$ *is simple for all* $\eta_n \in S$, $1 \leq n \leq N$.

To this end, we require the following lemma.

Lemma 2.21. *Let* $N \in \mathbb{N}$. *Let* X *be a normed linear space over* $\mathbb{K}$ ($\mathbb{R}$ *or* $\mathbb{C}$) *and let* $x_1, x_2, \dots, x_N$ *be non-zero elements of* X. *Then there exists an* $x^* \in X^*$ *such that* $\forall\ n = 1, 2, \dots, N$, $\langle x^*, x_n \rangle \neq 0$.

Proof. Consider the finite dimensional subspace F of X spanned by $x_1, x_2, \dots, x_N$. For each $n = 1, 2, \dots, N$, let F^*_n denote the subspace of F^* containing $x^* \in F^*$ such that $\langle x^*, x_n \rangle = 0$. Then $F^* \neq \cup_{n=1}^N F^*_n$ since a vector space cannot be written as a finite union of its proper subspaces. Hence, there exists an $x^* \in F^*$ such that $x^* \notin \cup_{n=1}^N F^*_n$. Hence, for all $n = 1, 2, \dots, N$, $\langle x^*, x_n \rangle \neq 0$. Finally, extend x^* to X^* using the Hahn-Banach Theorem. □

Proof of Proposition 2.20. As a part of the proof of Lemma 2.8, we proved that for a given $m \in \mathbb{N}$ and $\eta_0 \in Y'$, there exists a t_0 positive such that for all $t \in (0, t_0]$, the Bloch eigenvalue $\lambda_m(\eta; A + tB)$ of the perturbed operator $\mathcal{A} + t\mathcal{B}$ is simple at η_0. In the present proposition, we shall make a Bloch eigenvalue $\lambda_m(\eta)$ of the operator $\mathcal{A}$ simple at a finite number of points in Y' through a perturbation in the coefficients.

As in the proof of Proposition 2.12, the perturbation at any $\eta_n \in S$ gives rise to a selfadjoint holomorphic family of type (B), real-analytic for $t \in (-\sigma_0, \sigma_0)$, where $\sigma_0 = \frac{\alpha}{2\|B\|_{L^\infty}}$. Suppose that the eigenvalue $\lambda_m(\eta_n)$ of the operator $\mathcal{A}(\eta_n)$ has multiplicity h_n. For the perturbed operator $-\nabla \cdot (A + tB)\nabla$, the eigenvalue $\lambda_m(\eta_n)$ splits into h_n branches. Suppose that the h_n eigenvalues and eigenvectors are given as follows. For $n = 1, 2, \dots, N$ and $r = 1, 2, \dots, h_n$ and $t \in (-\sigma_0, \sigma_0)$:

$$\lambda^r_m(t; \eta_n) = \lambda_m(\eta_n) + t a^r_m(\eta_n) + t^2 \beta^r_m(t, \eta_n)$$
$$u^r_m(t; \eta_n) = u^r_m(\eta_n) + t v^r_m(\eta_n) + t^2 w^r_m(t, \eta_n),$$

where $\beta^r_m(t, \eta_n)$ and $w^r_m(t, \eta_n)$ are real-analytic functions.

As in (2.1), the following system of equations holds true for $n = 1, 2, \ldots, N$ and $r = 1, 2, \ldots, h_n$:

$$\int_Y B(\nabla + i\eta_n)u_m^r(\eta_n) \cdot \overline{(\nabla + i\eta_n)u_m^s(\eta_n)}\ dy = a_m^r(\eta_n)\delta_{rs}.$$

The above equations define operators that act on the unperturbed eigenspaces at each η_n. The multiplicity would go down if we find B and bases for the unperturbed eigenspaces in which some off-diagonal entry, in particular, the $(1, 2)$-entry is non-zero. To achieve this, we proceed as in the proof of Proposition 2.12. For any choice of basis of the unperturbed eigenspace at η_n, we find that either (2.2) or (2.3) holds. However, we cannot use this idea anymore, since, different η_n would have different matrices B. To remedy this, we notice that, at each $\eta_n = (\eta_{n,1}, \eta_{n,2}, \ldots, \eta_{n,d})$, for a basis given by $\{f_n^1, f_n^2, \ldots, f_n^{h_n}\}$ either

$$\sum_{l=1}^{d}(\partial_l + i\eta_{n,l})f_n^1(\partial_l - i\eta_{n,l})\overline{f_n^2} \not\equiv 0, \tag{2.21}$$

or, if the above sum is zero, then in the modified basis $\{f_n^1, f_n^1 + f_n^2, f_n^3, \ldots, f_n^{h_n}\}$,

$$\sum_{l=1}^{d}(\partial_l + i\eta_{n,l})f_n^1(\partial_l - i\eta_{n,l})\overline{(f_n^1 + f_n^2)} = \sum_{l=1}^{d}|(\partial_l + i\eta_{n,l})f_n^1|^2 \not\equiv 0, \tag{2.22}$$

provided that $f_n^1 \neq \exp(-i\eta_n \cdot y)$.

We can always choose f_n^1 to be a function different from $\exp(-i\eta_n \cdot y)$ since at any of the η_n, we have an eigenspace of dimension greater than 1. For each η_n, either the real part or the imaginary part of the expressions in (2.21) and (2.22) is non-zero. We shall call the non-zero part as p_n. If both are non-zero, we may choose either one. This will make sure that we have a collection of only real-valued functions.

By the above procedure, we have N elements of $L^1_\sharp(Y, \mathbb{R})$, again labelled as $\{p_1, p_2, \ldots, p_N\}$. By Lemma 2.21, there is an $\Upsilon \in (L^1_\sharp(Y, \mathbb{R}))^*$ such that $\Upsilon(p_n) \neq 0$ for all $n = 1, 2, \ldots, N$. By duality, there exists a $\beta \in L^\infty_\sharp(Y, \mathbb{R})$ such that $\Upsilon(p_n) = \int_Y \beta p_n\ dy \neq 0$.

Define $B = diag(\beta, \beta, \ldots, \beta)$, then either

$$\operatorname{Re}\int_Y B(\nabla + i\eta_n)f_n^1 \cdot \overline{(\nabla + i\eta_n)f_n^2}\ dy \neq 0,$$

or

$$\operatorname{Re}\int_Y B(\nabla + i\eta_n)f_n^1 \cdot (\nabla - i\eta_n)(\overline{f_n^1} + \overline{f_n^2})\ dy \neq 0,$$

depending on η_n.

At the end of this step, the multiplicity of $\lambda_m(\eta)$ at each of the points η_n will reduce at least by 1. We repeat the procedure with the points among $\{\eta_1, \eta_2, \ldots, \eta_N\}$ where the eigenvalue is still multiple. Finally, we require at most M steps to make the Bloch eigenvalue simple at each of these points, where $M = \max_{1\le n\le N} h_n$. □

Lemma 2.22. *Let $N \in \mathbb{N}$. Let $A \in M_B^>$ and let B be a real symmetric matrix with $L^\infty_\sharp(Y)$-entries. Let $\mathcal{A} = -\nabla \cdot (A\nabla)$ be a periodic elliptic differential operator. Let λ_0 be the upper endpoint of a spectral gap, which is attained by the Bloch eigenvalue $\lambda_m(\eta)$ at finitely many points $\eta_1, \eta_2, \ldots, \eta_N$ in Y'. Then for all $n \in \{1, 2, \ldots, N\}$, there is a neighborhood $\mathcal{O}_n$ of η_n such that for all $t \in (0, t_0]$, the m^{th} Bloch eigenvalue $\lambda_m(\eta, A+tB)$ is simple for all $\eta \in \mathcal{O}_n$ where the spectral edge is attained.*

Proof. We know that if a Bloch eigenvalue is simple at a point $\eta \in Y'$, it continues to remain simple in its neighborhood. The same remark applies here, i.e., for $n = 1, 2, \ldots, N$, a neighborhood exists in which the perturbed Bloch eigenvalue is simple. However, it is not clear that these neighborhoods can be chosen independently of $t \in (0, t_0]$. Following [PS17], we shall prove that for each $\{\eta_n\}_{n=1}^N$, there is a neighborhood, not depending on t, in which the m^{th} Bloch eigenvalue is simple at all points in it where the spectral edge is attained. We will indicate here how this may be achieved in the case of a single point η_0 of multiplicity 2. The spectral edges of multiplicity greater than 2 are handled in a recursive manner and the spectral edges at multiple points are handled by making using of Proposition 2.20, where a common perturbation B is obtained for finitely many points of multiplicity.

By the construction in Lemma 2.18, there is a neighborhood $\mathcal{G}$ of η_0 in which the multiplicity of the eigenvalue $\lambda_m(\eta)$ does not exceed 2. Further by (L2), there is an orthonormal set of functions $\{\phi_1(\eta, A + tB), \phi_2(\eta, A + tB)\}$ real-analytic for $\eta \in \mathcal{G}$ and sufficiently small t, such that the linear subspace $V(\eta, t)$ generated by these functions is the direct sum of the eigenspaces corresponding to the eigenvalues $\lambda_m(\eta, A + tB)$ and $\lambda_{m+1}(\eta, A + tB)$. Finally, in (L3), we show that we may choose these functions such that $\langle \phi_j, \dot{\phi}_k\rangle = 0$ for $j, k = 1, 2$.

As a consequence, we can write a 2×2 matrix C, with real-analytic entries whose eigenvalues coincide with $\lambda_m(\eta, A + tB)$ and $\lambda_{m+1}(\eta, A + tB)$ in a neighbourhood of $\eta = \eta_0$ and $t = 0$. Consider the sesquilinear form

$$\tilde{a}(\eta, t)(u, v) := \int_Y (A + tB)(\nabla + i\eta)u \cdot (\nabla - i\eta)\bar{v}\, dy,$$

then the entries of $C = (c_{jk})$ are given by

$$c_{jk}(\eta, t) = \tilde{a}(\eta, t)\left(\phi_j(\eta, A + tB), \phi_k(\eta, A + tB)\right).$$

For $r = 1, 2$, we may write

$$\phi_r(\eta, A + tB) = \psi_r(\eta) + t\nu_j(\eta) + O(t^2),$$

where the order is uniform for $\eta \in \mathcal{G}$ due to the analyticity of ϕ_r with respect to η and t. For a given basis $\Phi = \{u_1, u_2\}$ of the 2-dimensional subspace $V(\eta, t)$, define

$$b_{jk}(\eta, \Phi) := \int_Y B(\nabla + i\eta)u_j \cdot (\nabla - i\eta)\overline{u}_k \, dy.$$

Also, denote by $\Psi(\eta)$ the basis $\{\psi_1(\eta), \psi_2(\eta)\}$ of $V(\eta, 0)$. Then the entries of matrix C may be written as

$$c_{jk}(\eta, t) = \tilde{a}(\eta, 0)(\psi_j(\eta), \psi_k(\eta)) + t\, b_{jk}(\eta, \Psi(\eta)) + O(t^2).$$

By Proposition 2.20, there is a basis $\Theta = \{\theta_1, \theta_2\}$ of $V(\eta_0, 0)$ such that $b_{jk}(\eta_0, \Theta) \neq 0$. Without loss of generality, we may take $b_{jk}(\eta_0, \Theta) = 1$. By analysis of the discriminant of the matrix $\{b_{jk}(\eta_0, \Theta)\}_{j,k=1}^2$, we may conclude that for any other basis of $V(\eta_0, 0)$, and in particular for the basis $\Psi(\eta_0) = \{\psi_1(\eta_0), \psi_2(\eta_0)\}$, we either have $|b_{12}(\eta_0, \Psi(\eta_0))| \geq 1/4$ or $|b_{11}(\eta_0, \Psi(\eta_0)) - b_{22}(\eta_0, \Psi(\eta_0))| \geq 1$. Due to analyticity of $\{\psi_1(\eta), \psi_2(\eta)\}$, the matrix $\{b_{jk}(\eta, \Psi(\eta))\}_{j,k=1}^2$ is close to the matrix $\{b_{jk}(\eta_0, \Psi(\eta_0))\}_{j,k=1}^2$ whenever η is close to η_0. Hence, we may conclude that there is a neighborhood $\tilde{\mathcal{G}}$ of η_0 such that for all $\eta \in \tilde{\mathcal{G}}$, we either have $|b_{12}(\eta, \Psi(\eta))| \geq 1/8$ or $|b_{11}(\eta, \Psi(\eta)) - b_{22}(\eta, \Psi(\eta))| \geq 3/4$. We will now prove that in either of these cases, the points in $\tilde{\mathcal{G}}$ where the spectral edge is attained are simple.

If $\eta \in \tilde{\mathcal{G}}$ is such that $|b_{12}(\eta, \Psi(\eta))| \geq 1/8$, then C has distinct eigenvalues. On the other hand, suppose that $\eta \in \tilde{\mathcal{G}}$ is such that $|b_{12}(\eta, \Psi(\eta))| < 1/8$ but $|b_{11}(\eta, \Psi(\eta)) - b_{22}(\eta, \Psi(\eta))| \geq 3/4$. Without loss of generality and for convenience, we may assume that $b_{11}(\eta, \Psi(\eta)) \geq b_{22}(\eta, \Psi(\eta)) + 3/4$ and that $\lambda_m(\eta_0) = \lambda_{m+1}(\eta_0) = 0$. Further, at the cost of reducing the neighborhood $\tilde{\mathcal{G}}$ of η_0, we may take $|b_{jj}(\eta, \Psi(\eta)) - b_{jj}(\eta_0, \Psi(\eta_0))| \leq 1/8$ for $j = 1, 2$. Then the entries of matrix C have the form $c_{jj}(\eta_0, t) = t\, b_{jj}(\eta_0, \Psi(\eta_0)) + O(t^2)$ and $c_{12}(\eta_0, t) = t\, b_{12}(\eta_0, \Psi(\eta_0)) + O(t^2)$. Now, suppose that for some t, we have a multiple eigenvalue at $\eta \in \tilde{\mathcal{G}}$. Therefore, $c_{11}(\eta, t) = c_{22}(\eta, t)$. Subtracting the expression for $c_{11}(\eta, t)$ from the expression for $c_{22}(\eta, t)$ and noting that $\tilde{a}(\eta, 0)(\psi_j(\eta), \psi_j(\eta)) \geq 0$, we obtain $\tilde{a}(\eta, 0)(\psi_2(\eta), \psi_2(\eta)) \geq 3t/4 + O(t^2)$. Thus,

$$\begin{aligned} c_{22}(\eta, t) &= \tilde{a}(\eta, 0)(\psi_2(\eta), \psi_2(\eta)) + t b_{22}(\eta, \Psi(\eta)) + O(t^2) \\ &\geq t(3/4 + b_{22}(\eta, \Psi(\eta)) + O(t^2) \\ &\geq t(5/8 + b_{22}(\eta_0, \Psi(\eta_0)) + O(t^2) \\ &> c_{22}(\eta_0, t). \end{aligned}$$

Hence, η is not a point of minimum. □

In the next lemma, we shall prove that a spectral edge does not move very far for small perturbations in the coefficients of the periodic operator $\mathcal{A}$. We shall denote the operator $-\nabla \cdot (A + tB)\nabla$ as $\mathcal{A} + t\mathcal{B}$, where $\mathcal{A} = -\nabla \cdot (A\nabla)$ and $\mathcal{B} = -\nabla \cdot (B\nabla)$. Let S_t denote the set of points at which the new spectral edge is attained, i.e.,

$$S_t := \{\eta \in Y' : \text{ The Bloch eigenvalue } \lambda_m(\eta; A + tB) \text{ attains spectral edge at } \eta\}.$$

Lemma 2.23. *Let* $N \in \mathbb{N}$. *Let* $A \in M_B^>$ *and let* B *be a real symmetric matrix with* $L^\infty_\sharp(Y)$*-entries. Let* $\mathcal{A} = -\nabla \cdot (A\nabla)$ *be a periodic elliptic differential operator. Let* λ_0 *be the upper endpoint of a spectral gap, which is attained by the Bloch eigenvalue* $\lambda_m(\eta)$ *at finitely many points* $\eta_1, \eta_2, \ldots, \eta_N$ *in* Y'. *Given a* δ *belonging to the open interval* $(0, 1)$, *there is a* t_0 *such that*

$$\textit{for } t \in (0, t_0], \qquad S_t \subset \bigcup_{j=1}^{N} B(\eta_j, \delta),$$

where for all $j = 1, 2, \ldots, N$, *we have* $B(\eta_j, \delta) := \{\eta \in Y' : |\eta - \eta_j| < \delta\}$.

Proof. We prove this lemma by contradiction. Assume that there is a $\delta \in (0, 1)$ and sequences (t_n) and (ξ_n) such that $t_n \to 0$ and $\xi_n \in S_{t_n}$ such that

$$\forall\ 1 \le j \le N, \qquad |\xi_n - \eta_j| \ge \delta. \tag{2.23}$$

Let $\lambda_0(A + tB)$ denote the spectral edge associated to the operator $\mathcal{A} + t\mathcal{B}$. The perturbed spectral edge satisfies the following inequality.

$$\begin{aligned} |\lambda_0(A) - \lambda_0(A + t_nB)| &= |\min_{\eta \in Y'} \lambda_m(\eta; A) - \min_{\eta \in Y'} \lambda_m(\eta; A + t_nB)| \\ &= | - \max_{\eta \in Y'}(-\lambda_m(\eta; A)) + \max_{\eta \in Y'}(-\lambda_m(\eta; A + t_nB))| \\ &\le \max_{\eta \in Y'} |\lambda_m(\eta; A) - \lambda_m(\eta; A + t_nB)| \\ &\le Ct_n. \end{aligned} \tag{2.24}$$

Since (ξ_n) is a bounded sequence in Y', a subsequence of (ξ_n) converges to $\hat{\xi}$, which we continue to denote by (ξ_n).

We shall prove that

$$\lambda_m(\xi_n; A + t_nB) \to \lambda_m(\hat{\xi}; A). \tag{2.25}$$

Observe that

$$\begin{aligned}
&\left|\int_Y (A + t_n B)(\nabla + i\xi_n)u(\nabla - i\xi_n)\bar{u}\,dy - \int_Y A(\nabla + i\hat{\xi})u(\nabla - i\hat{\xi})\bar{u}\,dy\right| \\
&\quad\le \left|\int_Y A(\nabla + i\xi_n)u(\nabla - i\xi_n)\bar{u}\,dy - \int_Y A(\nabla + i\hat{\xi})u(\nabla - i\hat{\xi})\bar{u}\,dy\right| \\
&\qquad\qquad + t_n\|B\|_{L^\infty_\sharp(Y)}\int_Y |(\nabla + i\xi_n)u|^2\,dy \\
&\quad\le C\left(\left|\xi_n - \hat{\xi}\right|\|\nabla u\|_{L^2_\sharp(Y)}\|u\|_{L^2_\sharp(Y)} + \left|\xi_n - \hat{\xi}\right|\|u\|^2_{L^2_\sharp(Y)}\right) \\
&\qquad\qquad + t_n\|B\|_{L^\infty_\sharp(Y)}\int_Y |(\nabla + i\xi_n)u|^2\,dy,
\end{aligned}$$

for some generic constant C, which may change from line to line.

Therefore,

$$\begin{aligned}
&\int_Y (A + t_n B)(\nabla + i\xi_n)u(\nabla - i\xi_n)\bar{u}\,dy \\
&\quad\le \int_Y A(\nabla + i\hat{\xi})u(\nabla - i\hat{\xi})\bar{u}\,dy \\
&\qquad + C\left(\left|\xi_n - \hat{\xi}\right|\|\nabla u\|_{L^2_\sharp(Y)}\|u\|_{L^2_\sharp(Y)} + \left|\xi_n - \hat{\xi}\right|\|u\|^2_{L^2_\sharp(Y)}\right) \\
&\qquad + t_n\|B\|_{L^\infty_\sharp(Y)}\int_Y |(\nabla + i\xi_n)u|^2\,dy
\end{aligned}$$

Divide throughout by $\|u\|^2_{L^2_\sharp(Y)}$ and apply the min-max principle to obtain the following inequality:

$$\lambda_m(\xi_n; A + t_n B) \le \lambda_m(\hat{\xi}; A) + C\left(\sqrt{\lambda_m(0; I)} + 1\right)\left|\xi_n - \hat{\xi}\right| + t_n\|B\|_{L^\infty_\sharp(Y)}|\lambda_m(\xi_n; I)|.$$

Similarly,

$$\lambda_m(\hat{\xi}; A) \le \lambda_m(\xi_n; A + t_n B) + C\left(\sqrt{\lambda_m(0; I)} + 1\right)\left|\xi_n - \hat{\xi}\right| + t_n\|B\|_{L^\infty_\sharp(Y)}|\lambda_m(\xi_n; I)|.$$

Therefore,

$$|\lambda_m(\xi_n; A + t_n B) - \lambda_m(\hat{\xi}; A)| \le C\left(\sqrt{\lambda_m(0; I)} + 1\right)\left|\xi_n - \hat{\xi}\right| + t_n\|B\|_{L^\infty_\sharp(Y)}|\lambda_m(\xi_n; I)|. \tag{2.26}$$

In order to establish (2.25), notice that the first and second terms on RHS of (2.26) converge to 0 by the convergence of ξ_n to $\hat{\xi}$ and the boundedness of $\lambda_m(\xi_n; I)$ along with convergence of t_n to 0, respectively.

It follows from (2.24) and (2.25) that $\lambda_0(A) = \lambda_m(\hat{\xi}; A)$ and hence, $\hat{\xi}$ is also a spectral edge. By (2.23), this contradicts the initial assumption that there are only N points at which the spectral edge is attained. □

Proof of Theorem 2.5. The spectral edge of the operator $\mathcal{A}$ is attained at finitely many points $\eta_1, \eta_2, \ldots, \eta_N$ in Y'. Now, apply Lemma 2.22 to these points, so that there is a neighborhood $\mathcal{O}_j$ of each of the points $(\eta_j)_{j=1}^N$ in which any spectral edge is simple for all t, for a range of t. Each of these neighborhoods contain a ball, $B(\eta_j, \delta_j)$ of radius δ_j centered at η_j. Let $\delta := \min_{1 \le j \le N} \delta_j$, then by Lemma 2.23, there exists a t_0 positive such that for all $t \in (0, t_0]$, the spectral edge of the perturbed operator $\mathcal{A} + t\mathcal{B}$ is contained in the union of the balls $\bigcup_{j=1}^{N} B(\eta_j, \delta)$.

Hence, we have obtained a perturbation of the operator $\mathcal{A}$ such that its spectral edge is simple. □

2.7 Comments

In Section 1.4, we defined regularity of spectral edges as a collection of three conditions (R1), (R2), (R3). Our results show that these conditions may also interact. This thesis is only concerned with simplicity of spectral edges under a second order perturbation. However, the genericity of discreteness of spectral edges for nonsmooth second order elliptic operators is unexplored. The result of Filonov and Kachkovskiy [FK18], in particular, makes the assumption of C^2 second order coefficients which is not suitable for homogenization. It is an open question whether a second order elliptic operator with only measurable and bounded coefficients has an isolated spectral edge, generically or otherwise. On the other hand, non-degeneracy of spectral edges of Schrödinger operators under a perturbation of the potential was first studied in the work of Parnovski and Shterenberg [PS17]. We feel that their method should also extend to second order perturbations.

In proving the genericity theorems of this chapter, our perturbation often comes from Hahn-Banach theorem hence it is not clear whether it is constructible. However, the method of Parnovski and Shterenberg [PS17] assures us that a perturbation of a larger period may work.

Another possible extension of our work pertains to elliptic systems or to infinite periodic quantum graphs.

The bottom spectral edge of an elliptic system can only be made simple through a perturbation in lower order terms since a purely second order operator would have multiplicity of the bottom spectral edge corresponding to the number of equations in the system, for example, the elasticity operator has a multiplicity 3 bottom spectral edge. A lower order perturbation also has advantages in terms of regularity since a measurable and bounded perturbation would suffice unlike our theorem where $W^{1,\infty}$-regularity is required. Indeed, we would like to see if a lower order perturbation can be usefully employed for Bloch wave homogenization in the case of systems. It may be recalled that the ground states of systems of partial differential equations exhibit eigenvalue crossings and may not have an analytic description, which may cause difficulties in applications. Previously, Bloch wave homogenization for systems such as the elasticity system [SGV05], the Stokes system [ACFO07, AGV17] have been achieved through one-parameter perturbation theory and directional analyticity of the ground state.

On the other hand, infinite quantum graphs display a larger variety of phenomena. Parnovski and Shterenberg [PS17] furnish an example of a quantum graph with a non-isolated spectral edge. There are a number of works dealing with spectral properties of infinite quantum graphs [Car12, EKMN18, KN19]. We would like to see if some of our methods may be useful in this context.

Chapter 3

Application to Internal Edge Homogenization

We establish Bloch wave homogenization at an internal edge in the presence of multiplicity by employing a perturbation in the coefficients. We show that all the crossing Bloch modes contribute to the homogenization at the internal edge and that higher and lower modes do not contribute to the homogenization process.

3.1 Introduction

Birman and Suslina [BS04] have described homogenization as a *spectral threshold effect*. Their analysis focuses on finding norm resolvent estimates of different orders. For the operator $\mathcal{A}$, it is known that $\inf \sigma(\mathcal{A}) = 0$. This corresponds to the bottom edge of its spectrum. A non-zero spectral edge is called an internal edge. The notion of homogenization has been extended to internal edges in [Bir04, BS06]. Correctors for internal edge homogenization are developed in [SK09, SK11]. The aim of this chapter is to extend the internal edge homogenization theorem to a multiple spectral edge. To this end, we make use of the results in Chapter 2 to render a multiple spectral edge simple by perturbation in coefficients. Norm resolvent convergence of operators is quantified in terms of the convergence of coefficients in Lemma 3.1, whereas the Bloch wave homogenization of the perturbed operator is performed in Lemma 3.2. The contents of this chapter appear as part of the paper [1] which has been accepted for publication in Asymptotic Analysis.

3.2 Internal Edge Homogenization for a multiple spectral edge

In Chapter 1, Section 1.5, we reviewed the internal edge homogenization theorem of Birman and Suslina [BS06]. In this section, we shall prove a theorem corresponding to internal edge homogenization of the operator $\mathcal{A}^\epsilon = -\nabla \cdot \left(A(\frac{x}{\epsilon})\nabla\right)$ in $L^2(\mathbb{R}^d)$ for periodic A in the presence of multiplicity. We shall interpret the three assumptions (B1), (B2), (B3) that have been made on the spectral edge as hypotheses on the shape and structure of the spectral edge. Without knowledge of the shape and structure of the spectral edge, it is not possible to obtain any explicit homogenization result.

Starting with a spectral edge which is not simple, we shall appeal to Theorem 2.5 to modify the spectral edge so that it becomes simple. We shall make the following assumptions on the spectral edge. We assume the finiteness of the number of points at which the spectral edge is attained, however, since the contributions from different points are added up, we may as well assume that the spectral edge is attained at one point. Therefore, suppose that for the operator (1.1), a spectral gap exists. Let λ_0 denote the upper endpoint of this spectral gap of $\mathcal{A}$ and let m be the smallest index such that the Bloch eigenvalue λ_m attains λ_0, then $\lambda_0 = \min_{\eta\in Y'} \lambda_m(\eta)$.

Suppose that the spectral edge is attained at a unique point $\eta_0 \in Y'$. Also suppose that the eigenvalue λ_0 has multiplicity 2. Now, a perturbation matrix B with $L^\infty_\sharp(Y, \mathbb{R})$ entries, as in Theorem 2.5, is applied to the coefficients of operator $\mathcal{A}$, so that the new operator $\tilde{\mathcal{A}}(t) = \mathcal{A} + t\mathcal{B}$, has a simple spectral edge $\tilde{\lambda}_0(t)$ for sufficiently small t. However, the perturbed Bloch eigenvalues $\tilde{\lambda}_m(\eta, t)$ and $\tilde{\lambda}_{m+1}(\eta, t)$ are simple in some neighborhood $\mathcal{O}$ of η_0 for small enough t. The neighbourhood $\mathcal{O}$ is assumed to be independent of t.

For the perturbed spectral edge, we assume the following hypothesis

(C1) $\tilde{\lambda}_m(\eta; t)$ attains minimum $\tilde{\lambda}_0(t)$ at a unique point $\eta_0(t) \in \mathcal{O}$ and is non-degenerate on $\mathcal{O}$, i.e.,

$$\tilde{\lambda}_m(\eta; t) - \tilde{\lambda}_0(t) = (\eta - \eta_0(t))^T \tilde{B}_0(t)(\eta - \eta_0(t)) + O(|\eta - \eta_0(t)|^3),$$

for $\eta \in \mathcal{O}$, where $\tilde{B}_0(t)$ is positive definite, i.e., there is $\alpha_0 > 0$, independent of t, such that $\tilde{B}_0(t) > \alpha_0 I$. Further, the order above holds uniformly for sufficiently small t.

(C2) $\tilde{\lambda}_{m+1}(\eta;t)$ attains minimum $\tilde{\lambda}_1(t)$ at a unique point $\eta_1(t) \in \mathcal{O}$ and is non-degenerate on $\mathcal{O}$, i.e.,

$$\tilde{\lambda}_{m+1}(\eta;t) - \tilde{\lambda}_1(t) = (\eta - \eta_1(t))^T \tilde{B}_1(t)(\eta - \eta_1(t)) + O(|\eta - \eta_1(t)|^3),$$

for $\eta \in \mathcal{O}$, where $\tilde{B}_1(t)$ is positive definite, i.e., there is $\alpha_1 > 0$, independent of t, such that $\tilde{B}_1(t) > \alpha_1 I$. Further, the order above holds uniformly for sufficiently small t.

In essence, we are asking for the Bloch eigenvalues to have the shapes before and after the perturbation as in Fig. 3.1 and Fig. 3.2.

We will now set up notation for the internal edge homogenization theorem that we intend to prove. For $j = 0, 1$, let $\tilde{\psi}_{m+j}(y, \eta_j(t)) = \exp(iy \cdot \eta_j(t))\tilde{\phi}_{m+j}(y;t)$, where $\tilde{\phi}_{m+j}$ is a normalized eigenvector corresponding to the eigenvalue $\tilde{\lambda}_j(t) = \tilde{\lambda}_{m+j}(\eta_j(t))$ of $\tilde{\mathcal{A}}(\eta_j;t) = -(\nabla + i\eta_j)\cdot(A + tB)(\nabla + i\eta_j)$. In what follows, we shall choose $t = O(\epsilon^4)$. Define the following operators

$$R(\epsilon) := \left(\mathcal{A}^\epsilon - (\epsilon^{-2}\lambda_0 - \vartheta^2)I\right)^{-1}, \text{ and} \tag{3.1}$$

$$\begin{aligned}\tilde{R}^0(\epsilon) := {} & |Y|[\tilde{\psi}^\epsilon_m]\left(-\nabla \cdot \tilde{B}_0(t)\nabla + \vartheta^2 I\right)^{-1}[\overline{\tilde{\psi}^\epsilon_m}] \\ & + |Y|[\tilde{\psi}^\epsilon_{m+1}]\left(-\nabla \cdot \tilde{B}_1(t)\nabla + \vartheta^2 I\right)^{-1}[\overline{\tilde{\psi}^\epsilon_{m+1}}].\end{aligned} \tag{3.2}$$

We shall require the following two lemmas.

Lemma 3.1. *Let*

$$\tilde{R}(\epsilon) := \left(\tilde{\mathcal{A}}^\epsilon(t) - (\epsilon^{-2}\tilde{\lambda}_0(t) - \vartheta^2)I\right)^{-1}, \tag{3.3}$$

where $\tilde{\mathcal{A}}^\epsilon(t) = -\nabla \cdot \left(A(\frac{x}{\epsilon}) + tB(\frac{x}{\epsilon})\right)\nabla$ *is an unbounded operator in* $L^2(\mathbb{R}^d)$*, satisfying assumptions (C1) and (C2). Choose* $t = O(\epsilon^4)$*. Then*

$$||R(\epsilon) - \tilde{R}(\epsilon)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(\epsilon) \quad \text{as} \quad \epsilon \to 0.$$

Lemma 3.2. *With the same notation as in Lemma 3.1, it holds that*

$$||\tilde{R}(\epsilon) - \tilde{R}^0(\epsilon)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(\epsilon) \quad \text{as} \quad \epsilon \to 0.$$

The proofs of these lemmas will be the content of Subsections 3.2.1 and 3.2.2. Now, we state the internal edge homogenization theorem for a multiple spectral edge.

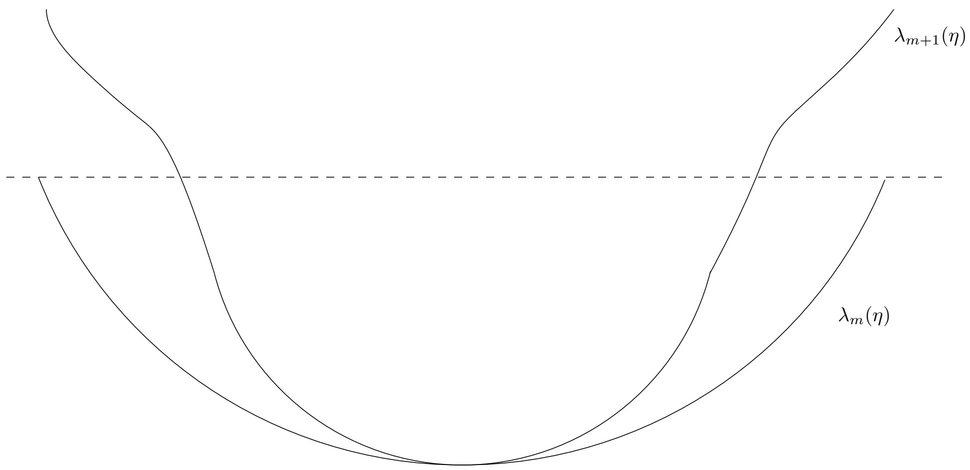


Figure 3.1: Spectral edge before perturbation.

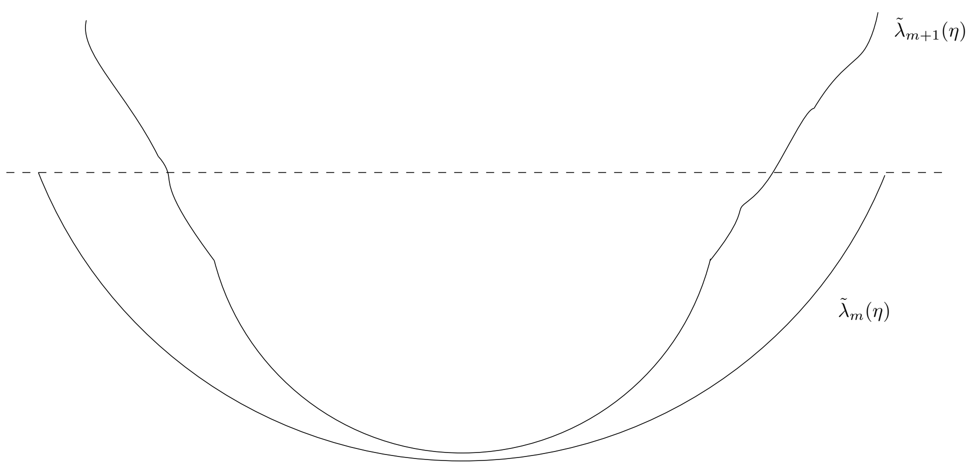


Figure 3.2: Spectral edge after perturbation.

Theorem 3.3. *Let $\mathcal{A}$ be the operator defined in $L^2(\mathbb{R}^d)$ as $\mathcal{A} := -\nabla \cdot (A\nabla)$. Suppose that the matrix A belongs to $M_B^{>}$. Let λ_0 be the upper edge of a spectral gap associated to operator $\mathcal{A}$. Suppose that λ_0 is attained at one point $\eta_0 \in Y'$ and its multiplicity is 2. Let $\vartheta^2 > 0$ be small enough so that $\lambda_0 - \vartheta^2$ remains in the spectral gap. Let $\mathcal{A}^\epsilon$ be defined as $\mathcal{A}^\epsilon = -\nabla \cdot \left(A(\frac{x}{\epsilon})\nabla\right)$ in $L^2(\mathbb{R}^d)$.*

Let $\tilde{\mathcal{A}}(t) = \mathcal{A} + t\mathcal{B}$ be a perturbation of $\mathcal{A}$ such that the perturbed operator has a simple spectral edge at $\tilde{\lambda}_0(t)$. Let $\tilde{\mathcal{A}}^\epsilon(t) = -\nabla \cdot \left(A(\frac{x}{\epsilon}) + tB(\frac{x}{\epsilon})\right)\nabla$. Choose $t = O(\epsilon^4)$. Assume conditions (C1), (C2) on the perturbed eigenvalues. Then

$$||R(\epsilon) - \tilde{R}^0(\epsilon)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(\epsilon) \quad as \quad \epsilon \to 0, \tag{3.4}$$

where $R(\epsilon)$ and $\tilde{R}^0(\epsilon)$ are defined in (3.1) and (3.2), respectively.

Proof of Theorem 3.3. Observe that

$$\begin{aligned} &||R(\epsilon) - \tilde{R}^0(\epsilon)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \\ &\quad \le ||R(\epsilon) - \tilde{R}(\epsilon)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} + ||\tilde{R}(\epsilon) - \tilde{R}^0(\epsilon)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}. \end{aligned} \tag{3.5}$$

Applying Lemmas 3.1 and 3.2 to (3.5), we obtain (3.4). □

Remark 3.4.

1. Theorem 3.3 allows the computation of the homogenized coefficients through perturbed Bloch eigenvalues. Both the crossing modes contribute to homogenization, even though the spectral edge is simple after the perturbation.

2. A perturbation of the form $\tilde{\mathcal{A}}(t)$, as mentioned in Theorem 3.3, exists for sufficiently small t by Theorem 2.5.

3. If the spectral edge is attained at finitely many points, the contribution to the effective operator from each of those points, are merely added up, as in Theorem 1.20. Hence, our assumption that the spectral edge is attained at one point is not restrictive. Further, the assumption that multiplicity of the spectral edge is 2 can also be relaxed, since our method allows successive reduction of multiplicity of Bloch eigenvalues at multiple points.

3.2.1 Proof of Lemma 3.1

The aim of this section is to prove Lemma 3.1. We begin by introducing some notation. Define the two resolvents $S(\epsilon)$ and $\tilde{S}(\epsilon)$ by

$$S(\epsilon) = \left(\mathcal{A} - (\lambda_0 - \epsilon^2\vartheta^2)I\right)^{-1} \quad \text{and} \quad \tilde{S}(\epsilon) = \left(\tilde{\mathcal{A}}(t) - (\tilde{\lambda}_0(t) - \epsilon^2\vartheta^2)I\right)^{-1} \tag{3.6}$$

Define

$$\mathfrak{h}[u] := \int_{\mathbb{R}^d} A\nabla u \cdot \nabla \overline{u}\, dy - \lambda_0 \int_{\mathbb{R}^d} |u|^2\, dy.$$

Then $\mathfrak{h}$ is a closed sectorial form with domain $H^1(\mathbb{R}^d)$.

Consider another form $\mathfrak{p}(t)$ with domain $H^1(\mathbb{R}^d)$ defined by

$$\mathfrak{p}(t)[u] := \int_{\mathbb{R}^d} tB\nabla u \cdot \nabla \overline{u}\, dy - (\tilde{\lambda}_0(t) - \lambda_0) \int_{\mathbb{R}^d} |u|^2\, dy.$$

To the forms $\mathfrak{h}$ and $\mathfrak{p}$, we shall apply the following theorem about continuity of resolvents which can be found in [Kat95, p. 340].

Theorem 3.5 [Kat95]. *Let $\mathfrak{h}$ be a densely defined, closed sectorial form bounded from below and let $\mathfrak{p}$ be a form relatively bounded with respect to $\mathfrak{h}$, so that $D(\mathfrak{h}) \subset D(\mathfrak{p})$ and*

$$|\mathfrak{p}[u]| \le a\|u\|^2 + b\mathfrak{h}[u], \tag{3.7}$$

where $0 \le b < 1$, but a may be positive, negative or zero. Then $\mathfrak{h} + \mathfrak{p}$ is sectorial and closed. Let H, K be the operators associated with $\mathfrak{h}$ and $\mathfrak{h} + \mathfrak{p}$, respectively. Let $\zeta \in \mathbb{C}$ not belong to the spectrum of H. Let $R(\zeta, H)$ denote the resolvent of H at ζ, i.e., $R(\zeta, H) = (H - \zeta I)^{-1}$. Also, suppose that

$$\|(a + bH)R(\zeta, H)\| < 1, \tag{3.8}$$

then ζ is not in the spectrum of K and

$$\|R(\zeta, K) - R(\zeta, H)\| \le \frac{4\|(a + bH)R(\zeta, H)\|}{(1 - \|(a + bH)R(\zeta, H)\|)^2}\|R(\zeta, H)\|. \tag{3.9}$$

□

In order to apply the theorem, we must verify the hypotheses (3.7) and (3.8). We shall prove that $\mathfrak{p}(t)$ is relatively bounded with respect to $\mathfrak{h}$, i.e., there exist $a, b \in \mathbb{R}$, such that:

$$|\mathfrak{p}(t)[u]| \le a\|u\|^2 + b\mathfrak{h}[u],$$

Observe that

$$\mathfrak{h}[u] \ge \alpha \int_{\mathbb{R}^d} |\nabla u|^2\, dy - \lambda_0 \int_{\mathbb{R}^d} |u|^2\, dy,$$

and

$$\begin{aligned}
&\mathfrak{p}(t)[u] \\
&\le t\|B\|_{L^\infty}\int_{\mathbb{R}^d}|\nabla u|^2\,dy + |\tilde{\lambda}_0(t)-\lambda_0|\int_{\mathbb{R}^d}|u|^2\,dy \\
&= \frac{t\|B\|_{L^\infty}}{\alpha}\left\{\int_{\mathbb{R}^d}\alpha|\nabla u|^2\,dy - \lambda_0\int_{\mathbb{R}^d}|u|^2\,dy\right\} + \left\{|\tilde{\lambda}_0(t)-\lambda_0| + \frac{t\|B\|_{L^\infty}}{\alpha}\lambda_0\right\}\int_{\mathbb{R}^d}|u|^2\,dy \\
&\le b\mathfrak{h}[u] + a\|u\|^2,
\end{aligned}$$

where $a = \left\{|\tilde{\lambda}_0(t)-\lambda_0| + \frac{t\|B\|_{L^\infty}}{\alpha}\lambda_0\right\} \le c_1 t$ and $b = \frac{t\|B\|_{L^\infty}}{\alpha} = c_2 t$ for some constants c_1 and c_2.

Next, observe that for selfadjoint operator H, the resolvent $R(\zeta, H)$ is a normal operator, therefore, we have (see [Kat95, p. 177])

$$\|R(\zeta, H)\| \le \frac{1}{\mathrm{dist}(\zeta, \sigma(H))}.$$

Further,

$$\begin{aligned}
\|(a+bH)R(\zeta,H)\| &\le \|aR(\zeta,H)\| + \|bHR(\zeta,H)\| \\
&\le \frac{a}{\mathrm{dist}(\zeta,\sigma(H))} + \|b(I+\zeta R(\zeta,H))\| \\
&\le \frac{a}{\mathrm{dist}(\zeta,\sigma(H))} + b\|I\| + b\|\zeta R(\zeta,H)\| \\
&\le \frac{a}{\mathrm{dist}(\zeta,\sigma(H))} + b + b\frac{|\zeta|}{\mathrm{dist}(\zeta,\sigma(H))}.
\end{aligned}$$

The operator corresponding to the sectorial form $\mathfrak{h}$ is $H := -\nabla\cdot A\nabla - \lambda_0 I$, therefore, $0 \in \sigma(H)$, so that, for $\zeta = -\epsilon^2\vartheta^2$, $\mathrm{dist}(\zeta, \sigma(H)) = \epsilon^2\vartheta^2$ for sufficiently small ϵ. Hence

$$\|(a+bH)R(\zeta,H)\| \le \frac{a}{\epsilon^2\vartheta^2} + 2b.$$

Notice that $R(\zeta, H) = S(\epsilon)$ and $R(\zeta, K) = \tilde{S}(\epsilon)$. Let us assume that t is small enough so that Theorem 3.5 can be applied to the resolvents in (3.6). In particular, we have

$$\begin{aligned}
\|S(\epsilon)-\tilde{S}(\epsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} &= \|R(\zeta,H)-R(\zeta,K)\| \\
&\le \frac{4(c_1 t + 2c_2 t\epsilon^2\vartheta^2)}{(\epsilon^2\vartheta^2 - c_1 t - 2c_2 t\epsilon^2\vartheta^2)^2}.
\end{aligned}$$

Choose t so that $c_1 t = \epsilon^4\vartheta^2$, then

$$\|S(\epsilon)-\tilde{S}(\epsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le \frac{4(1+2c_3\epsilon^2\vartheta^2)}{\vartheta^2(1-\epsilon^2-2c_3\epsilon^4\vartheta^2)^2}.$$

Further, for sufficiently small ϵ,

$$\|S(\epsilon) - \tilde{S}(\epsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le \frac{16(1 + c_3\vartheta^2)}{\vartheta^2(1 - c_3\vartheta^2)^2}. \tag{3.10}$$

Proof of Lemma 3.1. Define the scaling transformation T_ϵ by

$$T_\epsilon : u(y) \mapsto \epsilon^{d/2} u(\epsilon y).$$

These are unitary operators on $L^2(\mathbb{R}^d)$. For the operators (3.1) and (3.3), it holds that

$$R(\epsilon) = \epsilon^2 T_\epsilon^* S(\epsilon) T_\epsilon \quad \text{and} \quad \tilde{R}(\epsilon) = \epsilon^2 T_\epsilon^* \tilde{S}(\epsilon) T_\epsilon.$$

Proving Lemma 3.1 is equivalent to proving that

$$\|S(\epsilon) - \tilde{S}(\epsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O\left(\frac{1}{\epsilon}\right).$$

In fact, in (3.10), we proved

$$\|S(\epsilon) - \tilde{S}(\epsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(1).$$

□

3.2.2 Proof of Lemma 3.2

The aim of this section is to prove Lemma 3.2. Let $\left(\tilde{\lambda}_l(\eta;t)\right)_{l=1}^\infty$ and $\left(\tilde{\phi}_l(y,\eta;t)\right)_{l=1}^\infty$ be the Bloch eigenvalues and the corresponding orthonormal Bloch eigenvectors for the operator $\tilde{\mathcal{A}}(t)$, defined in Theorem 3.3. Let $\tilde{\psi}_l(y,\eta;t) = e^{iy\cdot\eta}\tilde{\phi}_l(y,\eta;t)$. In the sequel, we shall suppress the dependence on t for notational convenience. The operator $\tilde{\mathcal{A}}$ may be decomposed in terms of the Bloch eigenvalues as in the theorem below, a proof of which may be found in [BLP11].

Theorem 3.6. *Let* $g \in L^2(\mathbb{R}^d)$. *Define* l^{th} *Bloch coefficient of* g *as follows:*

$$(\tilde{\mathcal{B}}_l g)(\eta) = \int_{\mathbb{R}^d} \overline{\tilde{\psi}_l(y,\eta)} g(y)\ dy,\ \ l \in \mathbb{N}, \eta \in Y'.$$

Then the following inverse formula holds:

$$g(y) = \sum_{l=1}^\infty \int_{Y'} (\tilde{\mathcal{B}}_l g)(\eta)\tilde{\psi}_l(y,\eta)\ d\eta = \sum_{l=1}^\infty (\tilde{\mathcal{B}}_l^*)(\tilde{\mathcal{B}}_l g), \quad \textit{where}$$

$$(\tilde{\mathcal{B}}_l^* h)(y) = \int_{Y'} h(\eta)\tilde{\psi}_l(y,\eta)\, d\eta \quad \textit{for} \quad h \in L^2(Y').$$

In particular, the following representation holds for the operator $\tilde{\mathcal{A}}$:

$$\tilde{\mathcal{A}} = \sum_{l\in\mathbb{N}} \tilde{\mathcal{B}}_l^* \tilde{\lambda}_l \tilde{\mathcal{B}}_l.$$

Also,

$$R(\zeta, \tilde{\mathcal{A}}) = \left(\tilde{\mathcal{A}} - \zeta I\right)^{-1} = \sum_{l\in\mathbb{N}} \tilde{\mathcal{B}}_l^* (\tilde{\lambda}_l - \zeta)^{-1} \tilde{\mathcal{B}}_l.$$

□

Define the Fourier Transform and the inverse Fourier Transform

$$(\mathcal{F}u)(\eta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-iy\cdot\eta} u(y)\, dy, \qquad \left(\mathcal{F}^{-1}u\right)(\eta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{iy\cdot\eta} u(y)\, dy.$$

Proof of Lemma 3.2. Define the operator

$$\tilde{S}^0(\epsilon) := |Y|[\tilde{\psi}_m]\left(-\nabla\cdot\tilde{B}_0\nabla + \epsilon^2\vartheta^2 I\right)^{-1} [\overline{\tilde{\psi}_m}] + |Y|[\tilde{\psi}_{m+1}]\left(-\nabla\cdot\tilde{B}_1\nabla + \epsilon^2\vartheta^2 I\right)^{-1} [\overline{\tilde{\psi}_{m+1}}],$$

where $[\tilde{\psi}_m]$ denotes the operation of multiplication by the function $\tilde{\psi}_m(y, \eta_0(t))$ and $[\tilde{\psi}_{m+1}]$ denotes the operation of multiplication by the function $\tilde{\psi}_{m+1}(y, \eta_1(t))$.

By making $\mathcal{O}$ smaller if required (see (C1) and (C2)), we may assume that

$$2(\tilde{\lambda}_m(\eta) - \tilde{\lambda}_0) \geq (\eta - \eta_0)\cdot\tilde{B}_0(\eta - \eta_0),\ \eta \in \mathcal{O}, \quad \text{and}$$
$$2(\tilde{\lambda}_{m+1}(\eta) - \tilde{\lambda}_1) \geq (\eta - \eta_1)\cdot\tilde{B}_1(\eta - \eta_1),\ \eta \in \mathcal{O}.$$

Let χ be the characteristic function of $\mathcal{O}$, then the projections $F = \tilde{\mathcal{B}}_m^* \chi \tilde{\mathcal{B}}_m + \tilde{\mathcal{B}}_{m+1}^* \chi \tilde{\mathcal{B}}_{m+1}$ and $F^\perp = I - F$ commute with $\tilde{\mathcal{A}}$.

The operator $\tilde{S}^0(\epsilon)$ can be written as

$$\tilde{S}^0(\epsilon) = |Y|[\tilde{\phi}_m]\mathcal{F}^{-1}\left[\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1}\right]\mathcal{F}[\overline{\tilde{\phi}_m}]$$
$$+ |Y|[\tilde{\phi}_{m+1}]\mathcal{F}^{-1}\left[\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1}\right]\mathcal{F}[\overline{\tilde{\phi}_{m+1}}].$$

Next, we represent $\tilde{S}^0(\epsilon)$, as the sum of two terms: the first one is

$$\tilde{S}^0_\chi(\epsilon) = |Y|[\tilde{\phi}_m]\mathcal{F}^{-1}\left[\chi(\eta)\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1}\right]\mathcal{F}[\overline{\tilde{\phi}_m}]$$
$$+ |Y|[\tilde{\phi}_{m+1}]\mathcal{F}^{-1}\left[\chi(\eta)\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1}\right]\mathcal{F}[\overline{\tilde{\phi}_{m+1}}].$$

and the second one is

$$\tilde{S}^0_{1-\chi}(\epsilon) = |Y|[\tilde{\phi}_m]\mathcal{F}^{-1}\left[(1-\chi(\eta))\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1}\right]\mathcal{F}[\overline{\tilde{\phi}_m}]$$
$$+ |Y|[\tilde{\phi}_{m+1}]\mathcal{F}^{-1}\left[(1-\chi(\eta))\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1}\right]\mathcal{F}[\overline{\tilde{\phi}_{m+1}}]. \tag{3.11}$$

For the operators (3.2) and (3.3), it holds that

$$\tilde{R}(\epsilon) = \epsilon^2 T_\epsilon^* \tilde{S}(\epsilon) T_\epsilon \quad \text{and} \quad \tilde{R}^0(\epsilon) = \epsilon^2 T_\epsilon^* \tilde{S}^0(\epsilon) T_\epsilon.$$

Therefore, to prove Lemma 3.2, it is sufficient to prove that

$$\|\tilde{S}(\epsilon) - \tilde{S}^0(\epsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O\left(\frac{1}{\epsilon}\right). \tag{3.12}$$

Now, observe that

$$\begin{aligned}\|\tilde{S}(\epsilon) - \tilde{S}^0(\epsilon)\|_{L^2\to L^2} &= \|\tilde{S}(\epsilon)F^\perp + \tilde{S}(\epsilon)F - \tilde{S}^0_\chi(\epsilon) - \tilde{S}^0_{1-\chi}(\epsilon)\|_{L^2\to L^2} \\ &\le \|\tilde{S}(\epsilon)F^\perp\|_{L^2\to L^2} + \|\tilde{S}(\epsilon)F - \tilde{S}^0_\chi(\epsilon)\|_{L^2\to L^2} + \|\tilde{S}^0_{1-\chi}(\epsilon)\|_{L^2\to L^2}\end{aligned}$$

Thus, in order to prove (3.12), it is sufficient to prove the following:

$$\|\tilde{S}(\epsilon)F^\perp\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(1), \tag{3.13}$$

$$\|\tilde{S}^0_{1-\chi}(\epsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(1), \tag{3.14}$$

$$\|\tilde{S}(\epsilon)F - \tilde{S}^0_\chi(\epsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O\left(\frac{1}{\epsilon}\right). \tag{3.15}$$

Proof of (3.13): Notice that the Bloch wave decomposition of $\tilde{S}(\epsilon)$ is given by

$$\tilde{S}(\epsilon) = \sum_{l=1}^{\infty} \tilde{\mathcal{B}}_l^* \left(\tilde{\lambda}_l - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} \tilde{\mathcal{B}}_l.$$

We may write

$$\tilde{S}(\epsilon) = \tilde{S}(\epsilon)F + \tilde{S}(\epsilon)F^\perp,$$

where

$$\tilde{S}(\epsilon)F = \tilde{\mathcal{B}}_m^* \left(\tilde{\lambda}_m - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} \chi\tilde{\mathcal{B}}_m + \tilde{\mathcal{B}}_{m+1}^* \left(\tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} \chi\tilde{\mathcal{B}}_{m+1},$$

and

$$\begin{aligned}\tilde{S}(\epsilon)F^\perp = &\sum_{l\neq m,m+1} \tilde{\mathcal{B}}_l^* \left(\tilde{\lambda}_l - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} \tilde{\mathcal{B}}_l + \tilde{\mathcal{B}}_m^* \left(\tilde{\lambda}_m - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} (1-\chi)\,\tilde{\mathcal{B}}_m \\ &+ \tilde{\mathcal{B}}_{m+1}^* \left(\tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} (1-\chi)\,\tilde{\mathcal{B}}_{m+1}.\end{aligned} \tag{3.16}$$

To prove (3.13), notice that in the first term of (3.16), the sum does not include indices m and $m+1$, therefore, the Bloch eigenvalues $\tilde{\lambda}_l$ are bounded away from the spectral edge $\tilde{\lambda}_0$, uniformly in ϵ and hence, the expression $\left(\tilde{\lambda}_l - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1}$ is bounded independent of ϵ, for $l \neq m, m+1$. Due to the non-degeneracy conditions assumed in (C1) and (C2), the Bloch eigenvalues $\tilde{\lambda}_m$ and $\tilde{\lambda}_{m+1}$ are bounded away from $\tilde{\lambda}_0$ outside $\mathcal{O}$, independent of ϵ. Hence, the last two terms in (3.16) are bounded independent of ϵ.

Proof of (3.14): The proof of (3.14) follows from the positive-definiteness of $\tilde{B}_0$ and $\tilde{B}_1$ assumed in (C1) and (C2), which makes the operator norm of the terms in (3.11) independent of ϵ. Now, it only remains to prove (3.15).

Proof of (3.15): Write $\tilde{S}(\epsilon)F = S_0 + S_1$, where, for $j = 0, 1$,

$$\begin{aligned} S_j &:= \tilde{\mathcal{B}}^*_{m+j}\left(\tilde{\lambda}_{m+j} - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} \chi\, \tilde{\mathcal{B}}_{m+j} \\ &= X^*_{m+j}\left(\tilde{\lambda}_{m+j} - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} X_{m+j}, \end{aligned} \tag{3.17}$$

and, for $j = 0, 1$,

$$\begin{aligned} \left(X_{m+j}u\right)(\eta) &= \int_{\mathbb{R}^d} \chi(\eta)\overline{\tilde{\psi}_{m+j}}(y,\eta)u(y)\, dy \quad \text{and} \\ \left(X^*_{m+j}v\right)(y) &= \int_{Y'} \chi(\eta)\tilde{\psi}_{m+j}(y,\eta)v(\eta)\, d\eta. \end{aligned}$$

Write $\tilde{S}^0_\chi(\epsilon) = S^0_0 + S^0_1$, where, for $j = 0, 1$,

$$\begin{aligned} S^0_j &= |Y|[\tilde{\phi}_{m+j}]\mathcal{F}^{-1}\left((\eta - \eta_j)\cdot \tilde{B}_j(\eta - \eta_j) + \epsilon^2\vartheta^2 I\right)^{-1}(\chi)\,\mathcal{F}[\overline{\tilde{\phi}_{m+j}}] \\ &= (X^0_{m+j})^*\left((\eta - \eta_j)\cdot \tilde{B}_j(\eta - \eta_j) + \epsilon^2\vartheta^2 I\right)^{-1} X^0_{m+j}, \end{aligned} \tag{3.18}$$

and, for $j = 0, 1$,

$$\begin{aligned} \left(X^0_{m+j}u\right)(\eta) &= \int_{\mathbb{R}^d} \chi(\eta)e^{-iy\cdot\eta}\overline{\tilde{\phi}_{m+j}}(y,\eta_j)u(y)\, dy \\ \text{and} \qquad \left(X^0_{m+j}\right)^* v(y) &= \int_{\mathbb{R}^d} \chi(\eta)e^{iy\cdot\eta}\tilde{\phi}_{m+j}(y,\eta_j)v(\eta)\, d\eta. \end{aligned}$$

Observe that,

$$||\tilde{S}(\epsilon)F - \tilde{S}^0_\chi(\epsilon)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le ||S_0 - S^0_0||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} + ||S_1 - S^0_1||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}.$$

Therefore, to prove (3.15), it remains to prove that for $j = 0, 1$,

$$||S_j - S^0_j||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O\left(\frac{1}{\epsilon}\right).$$

where S_j, S^0_j are defined in (3.17), (3.18).

Consider

$$\begin{aligned} \epsilon||S_0 - S^0_0|| = \epsilon||X^*_m[\left(\tilde{\lambda}_m - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1}]X_m \\ - (X^0_m)^*\left((\eta - \eta_0)\cdot \tilde{B}_0(\eta - \eta_0) + \epsilon^2\vartheta^2 I\right)^{-1} X^0_m||. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\epsilon\|S_0 - S_0^0\| \le \epsilon\|X_m^*[\big(\tilde{\lambda}_m - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\big)^{-1}]X_m \\
- X_m^*\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1} X_m\| \\
+ \epsilon\|X_m^*\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1} X_m \\
- (X_m^0)^*\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1} X_m^0\|. \qquad (3.19)
\end{aligned}
$$

The first of the two terms on the right hand side (RHS) in the inequality (3.19) is estimated by using the following chain of inequalities:

$$
\begin{aligned}
&\epsilon\left|\left(\tilde{\lambda}_m - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} - \left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1}\right| \\
&\quad\le c\epsilon|\eta-\eta_0|^3\left(\tilde{\lambda}_m - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1}\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1} \\
&\quad\le c|\eta-\eta_0|^2\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0)\right)^{-1} 2\epsilon|\eta-\eta_0|\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1} \\
&\quad\le C_1.
\end{aligned}
$$

The second term on the RHS in inequality (3.19) may be written as $\epsilon\|(W_0^* W_0 - (W_0^0)^* W_0^0)\|$, where

$$
W_0 = \left[\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1/2}\right] X_m
$$

and

$$
W_0^0 = \left[\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1/2}\right] X_m^0.
$$

Therefore,

$$
W_0 - W_0^0 = \left[\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1/2}\right](X_m - X_m^0),
$$

where $X_m - X_m^0$ is the integral operator in $L^2(\mathbb{R}^d)$ defined by

$$
(X_m - X_m^0)(u)(\eta) = \int_{\mathbb{R}^d} \chi(\eta) e^{-iy\cdot\eta}\left(\overline{\tilde{\phi}_m}(y,\eta) - \overline{\tilde{\phi}_m}(y,\eta_0)\right) u(y)\, dy.
$$

Now, we shall analyze the boundedness of the integral operator defined above. To this end, we note that, due to simplicity of the Bloch eigenvalue $\tilde{\lambda}_m(\eta,t)$ in $\mathcal{O}$, the Bloch eigenfunction $\tilde{\phi}_m(y,\eta,t)$ is analytic $H^1_\sharp(Y)$-valued function with respect to $\eta \in \mathcal{O}$ and for small t. As a consequence, we may write, for $\eta \in \mathcal{O}$,

$$
\tilde{\phi}_m(y,\eta,t) - \tilde{\phi}_m(y,\eta_0,t) = \sum_{k=1}^{d} (\eta_k - \eta_{0,k})\gamma_k(y,\eta,t),
$$

where

$$\gamma_k(y,\eta,t) = \int_0^1 \frac{\partial\tilde{\phi}_m}{\partial\eta_k}(y,\eta_0 + s(\eta-\eta_0),t)\,ds. \tag{3.20}$$

Note that by choosing a smaller set if required, we may assume that the functions $\tilde{\phi}_m(y,\eta,t)$ and $\gamma_k(y,\eta,t)$ are analytic in a complex ball, which we shall denote by $\tilde{O}$.

The function $\tilde{\phi}_m(y,\eta,t)$ is a solution of the equation

$$-(\nabla + i\eta)\cdot(A+tB)(\nabla+i\eta)\tilde{\phi}_m(y,\eta,t) = \tilde{\lambda}_m(\eta,t)\tilde{\phi}_m(y,\eta,t), \tag{3.21}$$

for $\eta \in O$, $y \in Y$ and small t. In order to find an estimate for $\tilde{\phi}_m(y,\eta,t)$, we begin by separating the real and imaginary parts in the above equation. This gives rise to a system of two real equations with the same principal parts. In [LU68, Chapter 7, Theorem 2.1], estimates of the maximum modulus of solutions of such systems are obtained under Dirichlet conditions on the boundary. Such estimates are obtained for periodic boundary conditions in a similar manner. In particular, we have

$$\max_{y\in Y} |\tilde{\phi}_m(y,\eta,t)| \leq C_2,$$

where the constant C_2 depends on the the coercivity constant of the matrix $A+tB$ and its L^∞ bound, which are uniformly bounded in t for small t. As a consequence, the constant in this bound can be made independent of t. Further, for $\eta \in O$ and small t, the L^2 bound of $\tilde{\phi}_m(y,\eta,t)$ is bounded uniformly, therefore, we obtain

$$\max_{y,t} |\tilde{\phi}_m(y,\eta,t)| \leq C_3.$$

Similar bounds can be obtained for the derivatives of $\tilde{\phi}_m(y,\eta,t)$ with respect to η, by differentiating (3.21) with respect to η and applying [LU68, Chapter 7, Theorem 2.1] repeatedly. Finally, we obtain, for any $r \in \mathbb{N}$,

$$\sup_{y,t} \|\tilde{\phi}_m(y,\cdot,t)\|_{H^r(O)} < \infty. \tag{3.22}$$

The same bound is transferred to $\gamma_k(y,\eta,t)$ by virtue of (3.20). More details of this estimate may be found in [Bir97].

For $2r > d$, the function $\chi(\eta)\gamma_k(y,\eta,t)$ is a multiplier on the set of kernels of bounded integral operators in $L^2(\mathbb{R}^d)$, due to [BS77, Theorem 9.1]. Now, we may write

$$W_0 - W_0^0 = \sum_{k=1}^{d} \left[\left((\eta-\eta_0)\cdot\tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I \right)^{-1/2} \chi(\eta)(\eta_k - \eta_{0,k}) \right] U_k,$$

where U_k is the integral operator in $L^2(\mathbb{R}^d)$ defined by

$$(U_k u)(\eta) = \int_{\mathbb{R}^d} \chi(\eta) e^{-iy\cdot\eta} \overline{\gamma_k}(y,\eta,t) u(y)\, dy,$$

whose kernel differs from the kernel of the Fourier transform by the multiplicative factor $\chi(\eta)\overline{\gamma}_k(y,\eta,t)$, which is a multiplier, as mentioned earlier. Therefore, U_k is bounded. Further,

$$|\eta_k - \eta_{0,k}| \left((\eta-\eta_0)\cdot \tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1/2} \leq C_4.$$

Hence,

$$\|W_0 - W_0^0\|_{L^2\to L^2} \leq C_5. \tag{3.23}$$

Also, $\epsilon\left((\eta-\eta_0)\cdot \tilde{B}_0(\eta-\eta_0) + \epsilon^2\vartheta^2 I\right)^{-1/2} \leq \vartheta^{-1}$. Therefore, $\epsilon\left(\|W_0\|_{L^2\to L^2} + \|W_0^0\|_{L^2\to L^2}\right) \leq C_6$. Now, we can estimate the second term in the RHS of (3.19) as

$$\begin{aligned}\epsilon\|W_0^* W_0 - (W_0^0)^* W_0^0\| &= \epsilon\|W_0^*(W_0 - W_0^0) + (W_0 - W_0^0)^* W_0^0\| \\ &\leq \epsilon\|W_0\|\|W_0 - W_0^0\| + \epsilon\|W_0 - W_0^0\|\|W_0^0\| \leq C_7.\end{aligned}$$

Finally, consider

$$\begin{aligned}\epsilon\|S_1 - S_1^0\| = \epsilon\|X_{m+1}^*[&\left(\tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1}]X_{m+1} \\ &- (X_{m+1}^0)^*\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1} X_{m+1}^0\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\epsilon\|S_1 - S_1^0\| \leq \epsilon\|X_{m+1}^*[&\left(\tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1}]X_{m+1} \\ &- X_{m+1}^*\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1} X_{m+1}\| \\ &+ \epsilon\|X_{m+1}^*\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1} X_{m+1} \\ &- (X_{m+1}^0)^*\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1} X_{m+1}^0\|.\end{aligned} \tag{3.24}$$

The first of the two terms on RHS in inequality (3.24) is estimated by using the following chain of inequalities:

$$\begin{aligned}&\epsilon|\left(\tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2\vartheta^2\right)^{-1} - \left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1}| \\ &\quad\leq \epsilon|\left(\tilde{\lambda}_{m+1} - \tilde{\lambda}_1 + \epsilon^2\vartheta^2\right)^{-1} - \left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1}| \\ &\quad\leq c\epsilon|\eta-\eta_1|^3\left(\tilde{\lambda}_{m+1} - \tilde{\lambda}_1 + \epsilon^2\vartheta^2\right)^{-1}\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1} \\ &\quad\leq c|\eta-\eta_1|^2\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1)\right)^{-1} 2\epsilon|\eta-\eta_1|\left((\eta-\eta_1)\cdot\tilde{B}_1(\eta-\eta_1) + \epsilon^2\vartheta^2 I\right)^{-1} \\ &\quad\quad\leq C_8,\end{aligned}$$

where the first inequality follows since $\tilde{\lambda_0} \geq \tilde{\lambda_1}$.

The proof of the boundedness of the second term on RHS in inequality (3.24) is similar to that of the boundedness of the second term on RHS in inequality (3.19).

□

3.3 Comments

Internal edge homogenization requires information about the shape and structure of an internal spectral edge. This appears to be unknowable to a great extent or only known through empirical means. However we feel that it may be possible to bring about a classification of spectral edges in the fashion of normal forms for eigenvalue crossings due to Hagedorn [Hag92]. Such a classification may allow us to define homogenization results for internal spectral edges of all kinds.

This chapter is an application of the perturbation theory of spectral edges. We feel that perturbation theory of spectral edges may be applied to many other problems such as effective mass [AP05], Anderson localization [Ves02], diffractive geometric optics [APR11, APR13]. Indeed, any problem that demands an analysis of parameterized eigenvalue problems may benefit from these methods.

Chapter 4

Bloch wave approach to almost periodic homogenization

Bloch wave homogenization is a spectral method for obtaining effective coefficients for periodically heterogeneous media. This method hinges on the direct integral decomposition of periodic operators, which is not available in a suitable form for almost periodic operators. In particular, the notion of Bloch eigenvalues and eigenvectors does not exist for almost periodic operators. However, we are able to recover the homogenization result in this case, by employing a sequence of periodic approximations to almost periodic operators.

4.1 Introduction

The aim of this work is to extend the framework of Bloch wave method [CV97] to almost periodic media. Many microstructures beyond periodic occur in nature, such as amorphous solids like glass, motion of 2D electrons in a magnetic field [Hof76], quasicrystals [SBGC84], etc. The mixing together of two periodic media or an interface problem involving two different periodic media on the two sides of the interface [BLBL15] may be thought of as an almost periodic microstructure. Further, quasicrystals, which were discovered by Schechtman [SBGC84], are often modeled by taking projections of periodic media in higher dimensions [KD86]. Finally, dimers and polymers have also been modeled with almost periodic potentials [CdO02]. We may also note that the spectral theory of almost periodic operators is a well studied subject [PF92]. Although almost periodic media is completely deterministic, it serves as a bridge to stochastic descriptions of nature. A large variety of seemingly random natural phenomena can be explained through almost periodic structures [All83].

The first author to study the homogenization of highly oscillatory almost periodic media was Kozlov [Koz78]. Unlike periodic media, the cell problem for almost periodic media is posed on $\mathbb{R}^d$ and may not have solutions in the class of almost periodic functions. This was remedied by an abstract approach outlined in [OZ82, JKO94] where solutions to the corrector equation were sought without derivatives.

Bloch wave method relies on direct integral decomposition of periodic operators. For almost periodic operators, a direct integral decomposition is proposed in [BT81], however its fibers do not have compact resolvent which prevents us from defining Bloch eigenvalues for the almost periodic operator. To overcome this difficulty, we make use of periodic approximations, which are defined by a "restrict and periodize" operation, employed earlier by Bourgeat and Piatnitski [BP04] for stochastic homogenization.

Bloch wave method is a spectral method of homogenization. In particular, it relies on tools from representation theory for periodic operators [Mau68]. For definiteness, consider an operator in $L^2(\mathbb{R}^d)$ of the form

$$\mathcal{F}^\epsilon u := -\frac{\partial}{\partial x_k}\left(\kappa_{kl}\left(\frac{x}{\epsilon}\right)\frac{\partial u}{\partial x_l}\right),$$

where the coefficients are measurable bounded, periodic and symmetric. Let $\mathbb{T}^d$ denote the d-dimensional torus. Then the operator $\mathcal{F}^\epsilon$ is unitarily equivalent to a direct integral, given by

$$\int_{\frac{\mathbb{T}^d}{\epsilon}}^{\oplus} \mathcal{F}^\epsilon(\xi)d\xi, \tag{4.1}$$

where the fibers $\mathcal{F}^\epsilon(\xi)$ have compact resolvent and hence each fiber has a countable sequence of eigenvalues and eigenfunctions $\{\lambda_n^\epsilon(\xi), \phi_n^\epsilon(x,\xi)\}_{n\in\mathbb{N}}$, which are known as Bloch eigenvalues and eigenfunctions when considered as functions of $\xi \in \mathbb{T}^d/\epsilon$. Define l^{th} Bloch coefficient of u by

$$(\mathcal{B}_l^\epsilon u)(\xi) = \int_{\mathbb{R}^d} \overline{\phi_l^\epsilon(x,\xi)}u(x)e^{-ix\cdot\xi}\,dx,\ \ l \in \mathbb{N}.$$

Then as a consequence of the representation (4.1), the equation $\mathcal{F}^\epsilon u^\epsilon = f$, where $f \in L^2(\mathbb{R}^d)$, can be written as a cascade of equations in the Bloch space, viz.,

$$
\begin{aligned}
\lambda_1^\epsilon(\xi)\mathcal{B}_1^\epsilon(u^\epsilon)(\xi) &= \mathcal{B}_1^\epsilon(f)\\
\lambda_2^\epsilon(\xi)\mathcal{B}_2^\epsilon(u^\epsilon)(\xi) &= \mathcal{B}_2^\epsilon(f)\\
&\vdots\\
\lambda_l^\epsilon(\xi)\mathcal{B}_l^\epsilon(u^\epsilon)(\xi) &= \mathcal{B}_l^\epsilon(f)\\
&\vdots
\end{aligned}
$$

Homogenized equation can be recovered by passing to the limit in the first equation. The rest of the equations do not contribute to homogenization. It is evident that the representation (4.1) is crucial in this method.

For almost periodic operators, we introduce periodic approximations on cubes of side length $2\pi R$ which will add yet another parameter to the problem. We perform a Bloch wave analysis of the approximation and pass to the limit in Bloch space, first as $\epsilon \to 0$, followed by $R \to \infty$. We mention some of the interesting techniques employed in this chapter. The approximate Bloch spectral problems are posed on varying Hilbert spaces indexed by R. The approximate corrector and approximate homogenized tensors are obtained in terms of the first Bloch eigenvector and Bloch eigenvalue of the periodization. The homogenization limit is given a unified treatment by working in the Besicovitch space of almost periodic functions. We will prove a module containment result for the correctors which is of independent interest. The proof of homogenization theorem for almost periodic media (Theorem 4.8) is new to our knowledge.

In a previous work [1], we have considered perturbations of coefficients of an operator which make spectral edges simple. This work may also be thought of in the same vein. The almost periodic operator is expected to have a Cantor like spectrum [DFG19], and hence ill-defined spectral edges. Periodic approximations serve to regularize the spectral edges.

Bloch wave method has also been extended to other non-periodic media such as Hashin-Shtrikman structures [BCG+18]. A notion of Bloch-Taylor waves for aperiodic media has been introduced in [BG19] using regularized correctors.

The contents of this chapter form a section of the preprint [2].

4.1.1 Plan of Chapter

The notation and definitions that will be required in the text have already been introduced in Section 1.6 of Chapter 1. In Section 4.3, we will discuss two kinds of periodic approximations for almost periodic functions. In Section 4.4, Bloch wave analysis of periodic approximations is performed. In Section 4.5, we prove the homogenization result by first taking the limit $\epsilon \to 0$, followed by the limit $R \to \infty$ in the Bloch transform of the periodic approximations. In Section 4.6, we prove that the homogenized coefficients of the periodic approximations converge to those of the almost periodic operator. In Section 4.7, we prove that the higher Bloch modes do not contribute to the homogenization process.

4.2 Almost Periodic Differential Operators

Consider the almost periodic second-order elliptic operator in divergence form given by

$$\mathcal{A}u := -\mathrm{div}(A\nabla u) = -\frac{\partial}{\partial y_k}\left(a_{kl}(y)\frac{\partial u}{\partial y_l}\right), \tag{4.2}$$

where summation over repeated indices is assumed and the coefficients satisfy the following assumptions:

(D1) The coefficients $A = (a_{kl}(y))$ are continuous bounded real-valued almost periodic functions defined on $\mathbb{R}^d$. In other words, $a_{kl} \in AP(\mathbb{R}^d)$.

(D2) The matrix $A = (a_{kl})$ is symmetric, i.e., $a_{kl}(y) = a_{lk}(y)\ \forall\, y \in \mathbb{R}^d$.

(D3) Further, the matrix A is *coercive*, i.e., there exists an $\alpha > 0$ such that

$$\forall\, v \in \mathbb{R}^d \text{ and } a.e.\, y \in \mathbb{R}^d, \langle A(y)v, v\rangle \geq \alpha ||v||^2. \tag{4.3}$$

Let Ω be an open set in $\mathbb{R}^d$. We are interested in the homogenization of the following equation posed in $H^1(\Omega)$

$$\mathcal{A}^\epsilon u^\epsilon := -\frac{\partial}{\partial x_k}\left(a^\epsilon_{kl}(\epsilon)\frac{\partial u^\epsilon}{\partial x_l}\right) = f, \tag{4.4}$$

where $f \in L^2(\Omega)$ and $a^\epsilon_{kl}(\epsilon) := a_{kl}\left(\frac{x}{\epsilon}\right)$. Suppose that u^ϵ converges weakly to a limit $u \in H^1(\Omega)$. We shall prove in the course of this chapter that u satisfies an equation of the form

$$\mathcal{A}^* u := -\frac{\partial}{\partial x_k}\left(a^*_{kl}(x)\frac{\partial u}{\partial x_l}\right) = f, \tag{4.5}$$

and we also identify the coefficients a^*_{kl}. The assumption of symmetry is not essential for the purposes of homogenization since it is possible to define a dominant Bloch mode [SGV04] in the non-selfadjoint case.

Homogenization of almost periodic media was first carried out by Kozlov [Koz78] using quasiperiodic approximations. Subsequently, an abstract approach was given in [OZ82, JKO94] which is described in Subsection 1.6.2.

Some notation that we make use of, is listed below:

- We shall call a bounded continuous matrix-valued function A almost periodic if each of its entries is an almost periodic function.
- The notation $\lesssim$ is shorthand for $\leq$ with a multiplicative constant which does not depend on ϵ and R but may depend on the dimension d, L^∞ bound of A, the coercivity constant α, etc.
- The notation $\fint_G b(t)\,dt$ denotes the average $\frac{1}{|G|}\int_G b(t)\,dt$ of a function b over $G \subset \mathbb{R}^d$. Sometimes, the notation $\mathcal{M}_G(b)$ is also used.
- For $L > 0$, let Y_L denote the set $[-L\pi, L\pi)^d$.

4.3 Periodic Approximations of Almost Periodic Functions

Equation (4.4) is not amenable to a Bloch wave analysis due to non-periodicity of the coefficients. Hence, we shall introduce some periodic approximations to the coefficients of the operator (4.2). These periodic approximations follow the simple principle of "restrict and periodize". Given $f \in AP(\mathbb{R}^d)$, define

$$f^R(y) = f(y) \text{ for } y \in Y_R = [-R\pi, R\pi)^d, \tag{4.6}$$

and extend to the whole of $\mathbb{R}^d$ by periodization, i.e, $f^R(y + 2\pi R p) = f(y)$ for all $p \in \mathbb{Z}^d$. Hence, the periodic approximation so constructed belongs to $L^\infty_\sharp(Y_R)$.

The sequence f^R may not converge in $L^\infty(\mathbb{R}^d)$. In fact, the functions which can be written as a uniform limit of periodic functions are called as limit-periodic functions [LZ82] and they form a subclass of almost periodic functions. However, the sequence is convergent in $L^2_{loc}(\mathbb{R}^d)$ as well as uniformly on compact subsets of $\mathbb{R}^d$. It is unclear if the sequence f^R converges to f in $B^2(\mathbb{R}^d)$. We remark here that convergence in $L^2_{loc}(\mathbb{R}^d)$ does not imply convergence in $B^2(\mathbb{R}^d)$.

Remark 4.1. Another periodic approximation to almost periodic functions is constructed in [Shu78]. Roughly speaking, given a $f \in AP(\mathbb{R}^d)$, there is a sequence of numbers $(T_n)_{n\in\mathbb{N}}$ going to ∞ and a sequence of T_nY-periodic functions P_n such that $||u - P_n||_{\infty, T_n Y} \to 0$ as $n \to \infty$. The notation $||\cdot||_{\infty, T_n Y}$ implies that the L^∞ norm is taken over the cube $T_n Y = [-T_n\pi, T_n\pi)^d$. The proof of this theorem involves approximation of irrationals by rationals by Dirichlet's Approximation Theorem. Clearly, either of these approximations may be used for our purposes. Note that the approximations in [Shu78] have the advantage that they are smooth being trigonometric polynomials; however, the sequence $(T_n)_{n\in\mathbb{N}}$ cannot be chosen.

4.3.1 Periodic Approximations of Almost Periodic Operators

For $R > 0$, we denote by $A^R = (a^R_{kl}(y))^d_{k,l=1}$ the periodic approximation of $A = (a_{kl}(y))^d_{k,l=1}$ at level R, as explained in (4.6), i.e., for $1 \le k, l \le d$,

$$\begin{cases} a^R_{kl}(y) = a_{kl}(y) & \text{for } y \in Y_R \\ a^R_{kl}(y + 2\pi R p) = a_{kl}(y) & \text{for } p \in \mathbb{Z}^d \end{cases} \tag{4.7}$$

The following operator will serve as a periodic approximation to $\mathcal{A}$.

$$\mathcal{A}^R u := -\operatorname{div}(A^R \nabla u) = -\frac{\partial}{\partial y_k}\left(a^R_{kl}(y)\frac{\partial u}{\partial y_l}\right).$$

Such an approximation has been considered in [BP04].

4.4 Bloch wave Analysis for Periodic Approximations

In this section, we shall perform a Bloch wave analysis for the periodic approximations of the operator in (4.2). In particular, we shall study, for each fixed $R > 0$, the Bloch waves for the operators in $L^2(\mathbb{R}^d)$ given by

$$\mathcal{A}^R u := -\operatorname{div}(A^R \nabla u) = -\frac{\partial}{\partial y_k}\left(a^R_{kl}(y)\frac{\partial u}{\partial y_l}\right). \tag{4.8}$$

Let $Y'_R := \left[-\frac{1}{2R}, \frac{1}{2R}\right)^d$ denote a basic cell for the dual lattice corresponding to $2\pi R\mathbb{Z}^d$. The operator $\mathcal{A}^R$ can be written as the direct integral $\int_{Y'_R}^{\oplus} \mathcal{A}^R(\eta)\, d\eta$, where

$$\mathcal{A}^R(\eta) = e^{-i\eta\cdot y}\mathcal{A}^R e^{i\eta\cdot y} = -\left(\frac{\partial}{\partial y_k} + i\eta_k\right)a^R_{kl}(y)\left(\frac{\partial}{\partial y_l} + i\eta_l\right), \tag{4.9}$$

is an unbounded operator in $L^2_\sharp(Y_R)$. As a consequence, the spectrum of the operator $\mathcal{A}^R$ is the union of spectra of $\mathcal{A}^R(\eta)$ as η varies in Y'_R [RS78, p. 284]. It can be shown that the operators $A^R(\eta)$ have compact resolvent [BLP11]. Therefore, $\mathcal{A}^R(\eta)$ has a sequence of eigenvalues and eigenvectors

$$\eta \mapsto (\lambda^R_m(\eta), \phi^R_m(y;\eta)), m = 1, 2, \dots, \tag{4.10}$$

which are called Bloch eigenvalues and eigenvectors.

Remark 4.2. We shall choose $||\phi^R_1(\cdot;\eta)||_{L^2_\sharp(Y_R)} = R^{d/2}$ and $\phi^R_1(y;0) = \frac{1}{(2\pi)^{d/2}}$ $\forall\, R > 0$.

4.4.1 Bloch Decomposition of $L^2(\mathbb{R}^d)$

In this section, we shall state the theorem on decomposition of functions in $L^2(\mathbb{R}^d)$ using Bloch waves. We shall not go through the details of the proof, which may be found in [BLP11], [SGV04] and [SGV05].

Consider the unbounded operator defined in $L^2(\mathbb{R}^d)$

$$\mathcal{A}^{R,\epsilon}u := -\mathrm{div}(A^{R,\epsilon}\nabla u) = -\frac{\partial}{\partial x_k}\left(a^{R,\epsilon}_{kl}(x)\frac{\partial u}{\partial x_l}\right), \tag{4.11}$$

where $a^{R,\epsilon}_{kl}(x) := a^R_{kl}\left(\frac{x}{\epsilon}\right)$.

By homothecy, the Bloch eigenvalues and Bloch eigenvectors for the operator (4.11) are

$$\lambda^{R,\epsilon}_m(\xi) = \epsilon^{-2}\lambda^R_m(\epsilon\xi),\ \phi^{R,\epsilon}_m(x;\xi) = \phi^R_m\left(\frac{x}{\epsilon};\epsilon\xi\right), \tag{4.12}$$

where $\lambda^R_m(\eta)$ and $\phi^R_m(\eta)$ are defined in (4.10).

Theorem 4.3. *Let* $R > 0$. *Let* $g \in L^2(\mathbb{R}^d)$. *Define the* m^{th} *Bloch coefficient of* g *as*

$$\mathcal{B}^{R,\epsilon}_m g(\xi) := \int_{\mathbb{R}^d} g(x)e^{-ix\cdot\xi}\overline{\phi^{R,\epsilon}_m(x;\xi)}\,dx,\ \ m \in \mathbb{N},\ \ \xi \in \epsilon^{-1}Y'_R. \tag{4.13}$$

1. *The following inverse formula holds*

$$g(y) = \int_{\epsilon^{-1}Y'_R}\sum_{m=1}^{\infty}\mathcal{B}^{R,\epsilon}_m g(\xi)\phi^{R,\epsilon}_m(x;\xi)e^{ix\cdot\xi}\,d\xi. \tag{4.14}$$

2. **Parseval's identity**

$$||g||^2_{L^2(\mathbb{R}^d)} = \sum_{m=1}^{\infty}\int_{\epsilon^{-1}Y'_R}|\mathcal{B}^{R,\epsilon}_m g(\xi)|^2\,d\xi. \tag{4.15}$$

3. **Plancherel formula** *For* $f, g \in L^2(\mathbb{R}^d)$*, we have*

$$\int_{\mathbb{R}^d} f(y)\overline{g(y)}\,dy = \sum_{m=1}^{\infty} \int_{\epsilon^{-1}Y'_R} \mathcal{B}^{R,\epsilon}_m f(\xi)\overline{\mathcal{B}^{R,\epsilon}_m g(\xi)}\,d\xi. \tag{4.16}$$

4. **Bloch Decomposition in** $H^{-1}(\mathbb{R}^d)$ *For an element* $F = u_0(x) + \sum_{j=1}^{N} \frac{\partial u_j(x)}{\partial x_j}$ *of* $H^{-1}(\mathbb{R}^d)$*, the following limit exists in* $L^2(\epsilon^{-1}Y'_R)$*:*

$$\mathcal{B}^{R,\epsilon}_m F(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \left\{ u_0(x)\overline{\phi^{R,\epsilon}_m(x;\xi)} + i\sum_{j=1}^{N} \xi_j u_j(x)\overline{\phi^{R,\epsilon}_m(x;\xi)} \right\} dx$$
$$- \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \sum_{j=1}^{N} u_j(x) \frac{\partial \overline{\phi^{R,\epsilon}_m}}{\partial x_j}(x;\xi)\,dx. \tag{4.17}$$

The definition above is independent of the particular representative of F.

5. *Finally, for* $g \in D(\mathcal{A}^{R,\epsilon})$*,*

$$\mathcal{B}^{R,\epsilon}_m(\mathcal{A}^{R,\epsilon} g)(\xi) = \lambda^{R,\epsilon}_m(\xi)\mathcal{B}^{R,\epsilon}_m g(\xi). \tag{4.18}$$

4.4.2 Bloch Transform converges to Fourier Transform

The following lemma says that as $\epsilon \to 0$, the first Bloch coefficient of a function converges to its Fourier transform, which is defined as $\hat{u}(\xi) = \int_{\mathbb{R}^d} u(y)e^{-ix\cdot\xi}\,dy$. This is a consequence of the Lipschitz continuity of $\phi^R_1(y;\eta)$ in η close to $\eta = 0$, and the choice of normalization of the first Bloch eigenfunctions (See Remark 4.2). For a proof, see [CV97].

Lemma 4.4. *Let* $R > 0$*. Let* $g, g^\epsilon \in L^2(\mathbb{R}^d)$ *be such that the support of* g^ϵ *is contained in a fixed compact subset* $K \subset \mathbb{R}^d$*, independent of* ϵ*. If* g^ϵ *converges weakly to* g *in* $L^2(\mathbb{R}^d)$*, then we have* $\chi_{\epsilon^{-1}U_R}\mathcal{B}^{R,\epsilon}_1 g^\epsilon(\xi) \rightharpoonup \hat{g}(\xi)$ *in* $L^2(\mathbb{R}^d_\xi)$*-weak.*

4.4.3 Regularity Properties of Bloch eigenvalues and eigenvectors

In physical applications, the regularity properties of Bloch eigenvalues and eigenvectors with respect to the dual parameter $\eta \in Y'_R$ plays an important role, for example, see: [ACP+04], [AP05], [APR11]. It is a simple consequence of the Courant-Fischer minmax principle that Bloch eigenvalues are Lipschitz continuous in the dual parameter [CV97]. However, such limited regularity is usually not sufficient for our purposes. We require the following theorem about the behavior of the first Bloch eigenvalue and eigenvector in a neighborhood of $0 \in Y'_R$.

Theorem 4.5. *There is a neighborhood* $U_R := \{\eta \in Y'_R : |\eta| < \delta_R\}$*, where* δ_R *is a positive real number, such that the first Bloch eigenvalue* $\lambda_1^R(\eta)$ *is analytic for* $\eta \in U_R$ *and the first Bloch eigenvector* $\phi_1^R(\eta) \in H^1_\sharp(Y_R)$ *may be chosen to be analytic for* $\eta \in U_R$*.*

A proof of Theorem 4.5 that uses the notion of infinite-dimensional determinants can be found in [CV97]. Another proof that uses the Kato-Rellich Theorem [RS78], [Kat95] may be found in [SGV04].

Remark 4.6. The radius δ_R of the neighborhood U_R depends on the gap between the first and second Bloch eigenvalues of the operator $\mathcal{A}^R$. The limit operator of $\mathcal{A}^R$ is the almost periodic operator $\mathcal{A}$ which often has a Cantor-like spectrum [DFG19]. Hence, we expect the spectral gap to vanish in the limit $R \to \infty$. Therefore, the neighborhood U_R is expected to shrink to 0 in the limit $R \to \infty$.

4.4.4 Derivatives of the first Bloch eigenvalue and eigenfunction

In the theory of periodic homogenization [BLP11], homogenized coefficients are given in terms of solutions of the cell problem which is an equation posed on the basic periodic cell. For the R^{th} periodic approximation (4.8), we recall the cell problem and the homogenized coefficients below.

The homogenized coefficients for the R^{th} periodic approximation are given by:

$$a_{kl}^{R,*} = \frac{1}{|Y_R|}\int_{Y_R} a_{kl}^R(y)\,dy + \frac{1}{|Y_R|}\int_{Y_R} a_{kp}^R(y)\frac{\partial w^{R,l}}{\partial y_p}\,dy, \tag{4.19}$$

where $w^{R,p} \in H^1_\sharp(Y_R)$ satisfy the following cell problems for $1 \le p \le d$:

$$\mathcal{A}^R w^{R,p} = -\frac{\partial}{\partial y_k}\left(a_{kl}^R(y)\,\frac{\partial w^{R,p}}{\partial y_l}\right) = \frac{\partial a_{lp}^R}{\partial y_l}(y) \text{ in } Y_R. \tag{4.20}$$

The functions $w^{R,p}$ are called *correctors* and w^R is called *corrector field.* We recall that $\lambda_1^R(\eta)$ and $\phi_1^R(\eta)$ are analytic in $U_R \subset Y'_R$. The proof of the following theorem is standard and may be found in [CV97] or [SGV04].

Theorem 4.7. *The first Bloch eigenvalue and eigenfunction of the* R^{th} *periodic approximation* $\mathcal{A}^R$ *satisfy:*

1. $\lambda_1^R(0) = 0$.

2. *The eigenvalue* $\lambda_1^R(\eta)$ *has a critical point at* $\eta = 0$, *i.e.,*

$$\frac{\partial \lambda_1^R}{\partial \eta_s}(0) = 0, \forall s = 1, 2, \ldots, d. \tag{4.21}$$

3. *For* $s = 1, 2, \ldots, d$, *the derivative of the eigenvector* $(\partial \phi_1^R / \partial \eta_s)(0)$ *satisfies:* $(\partial \phi_1^R / \partial \eta_s)(y; 0) - i\phi_1^R(y; 0) w^{R,s}(y)$ *is a constant in* y.

4. *The Hessian of the first Bloch eigenvalue at* $\eta = 0$ *is twice the homogenized matrix* $a_{kl}^{R,*}$:

$$\frac{1}{2} \frac{\partial^2 \lambda_1^R}{\partial \eta_k \partial \eta_l}(0) = a_{kl}^{R,*}. \tag{4.22}$$

4.4.5 Boundedness of Corrector Field

We will show that the sequence $(\nabla w^{R,p})_{R>0}$ is bounded in $B^2(\mathbb{R}^d)$, independent of R. We know that for each $R > 0$ and $1 \leq p \leq d$, $\nabla w^{R,p} \in (L^2_\sharp(Y_R))^d \subset (B^2(\mathbb{R}^d))^d$ satisfies

$$\mathcal{M}\left(A^R \nabla w^{R,p} \nabla w^{R,p}\right) = -\sum_{l=1}^{d} \mathcal{M}\left(a_{lp}^R \frac{\partial w^{R,p}}{\partial y_l}\right) \tag{4.23}$$

Using the coercivity and boundedness of the matrix A, we obtain:

$$\alpha ||\nabla w^{R,p}||^2_{(L^2_\sharp(Y_R))^d} \leq C ||\nabla w^{R,p}||_{(L^2_\sharp(Y_R))^d}$$

From the last equation, we obtain the norm-boundedness of $(\nabla w^{R,p})$ in $(L^2_\sharp(Y_R))^d$ and hence in $(B^2(\mathbb{R}^d))^d$.

4.4.6 Boundedness of homogenized tensors

Due to the boundedness of derivatives of the correctors proved in Subsection 4.4.5, the sequence of numbers $a_{kl}^{R,*}$, defined in (4.19), is bounded independently of R. Further, it follows from the identification (4.22) that the sequence of numbers $\frac{1}{2} \frac{\partial^2 \lambda_1^R}{\partial \eta_k \partial \eta_l}(0)$ is bounded. Hence, there is a subsequence, still labeled by R, for which the sequence $\frac{1}{2} \frac{\partial^2 \lambda_1^R}{\partial \eta_k \partial \eta_l}(0)$ converges. We shall call this limit as a_{kl}^*, i.e.,

$$\lim_{R \to \infty} \frac{\partial^2 \lambda_1^R}{\partial \eta_k \partial \eta_l}(0) = 2a_{kl}^*. \tag{4.24}$$

4.5 Homogenization Result

In this section, we shall state the homogenization result for almost periodic media and prove it using the Bloch wave method. It will be seen in a further section that the coefficients a^*_{kl}, defined in (4.24), coincide with the homogenized coefficients for almost periodic media [OZ82]. In this section, we shall assume summation over repeated indices for ease of notation.

Theorem 4.8. *Let Ω be an arbitrary domain in $\mathbb{R}^d$ and $f \in L^2(\Omega)$. Let $u^\epsilon \in H^1(\Omega)$ be such that u^ϵ converges weakly to u^* in $H^1(\Omega)$, and*

$$\mathcal{A}^\epsilon u^\epsilon = f \text{ in } \Omega. \tag{4.25}$$

Then

1. *For all $k = 1, 2, \dots, d$, we have the following convergence of fluxes:*

$$a^\epsilon_{kl}(x)\frac{\partial u^\epsilon}{\partial x_l}(x) \rightharpoonup a^*_{kl}\frac{\partial u^*}{\partial x_l}(x) \text{ in } L^2(\Omega)\text{-weak.} \tag{4.26}$$

2. *The limit u^* satisfies the homogenized equation:*

$$\mathcal{A}^{\hom} u^* = -\frac{\partial}{\partial x_k}\left(a^*_{kl}\frac{\partial u^*}{\partial x_l}\right) = f \text{ in } \Omega, \tag{4.27}$$

where $(a^*_{kl})_{1 \le k,l \le d}$ are given in (4.24).

The proof of Theorem 4.8 is divided into the following steps. We begin by localizing the equation (4.25) which is posed on Ω, so that it is posed on $\mathbb{R}^d$. We take the first Bloch transform $\mathcal{B}^{R,\epsilon}_1$ of this equation and pass to the limit $\epsilon \to 0$, followed by the limit $R \to \infty$. The proof relies on the analyticity of the first Bloch eigenvalue and eigenfunction in a neighborhood of $0 \in Y'_R$. The limiting equation is an equation in Fourier space. The homogenized equation is obtained by taking the inverse Fourier transform.

4.5.1 Localization

Let ψ_0 be a fixed smooth function supported in a compact set $K \subset \mathbb{R}^d$. Since u^ϵ satisfies $\mathcal{A}^\epsilon u^\epsilon = f$, $\psi_0 u^\epsilon$ satisfies

$$\mathcal{A}^{R,\epsilon}(\psi_0 u^\epsilon)(x) = \psi_0 f(x) + g^\epsilon(x) + h^{R,\epsilon}(x) + l^{R,\epsilon}(x) \text{ in } \mathbb{R}^d, \tag{4.28}$$

where

$$g^\epsilon(x) := -\frac{\partial\psi_0}{\partial x_k}(x)a^\epsilon_{kl}(x)\frac{\partial u^\epsilon}{\partial x_l}(x), \tag{4.29}$$

$$h^{R,\epsilon}(x) := -\frac{\partial}{\partial x_k}\left(\frac{\partial\psi_0}{\partial x_l}(x)a^{R,\epsilon}_{kl}(x)u^\epsilon(x)\right), \tag{4.30}$$

$$l^{R,\epsilon}(x) := -\frac{\partial}{\partial x_k}\left(\psi_0(x)\left(a^{R,\epsilon}_{kl}(x) - a^\epsilon_{kl}(x)\right)\frac{\partial u^\epsilon}{\partial x_l}(x)\right). \tag{4.31}$$

While the sequence g^ϵ is bounded in $L^2(\mathbb{R}^d)$, the sequences $h^{R,\epsilon}$ and $l^{R,\epsilon}$ are bounded in $H^{-1}(\mathbb{R}^d)$. Taking the first Bloch transform of both sides of the equation (4.28), we obtain for $\xi \in \epsilon^{-1}U_R$ a.e.

$$\lambda^{R,\epsilon}_1(\xi)\mathcal{B}^{R,\epsilon}_1(\psi_0u^\epsilon)(\xi) = \mathcal{B}^{R,\epsilon}_1(\psi_0f)(\xi) + \mathcal{B}^{R,\epsilon}_1g^\epsilon(\xi) + \mathcal{B}^{R,\epsilon}_1h^{R,\epsilon}(\xi) + \mathcal{B}^{R,\epsilon}_1l^{R,\epsilon}(\xi) \tag{4.32}$$

We shall now pass to the limit $\epsilon \to 0$, followed by the limit $R \to \infty$ in the equation (4.32).

4.5.2 Limit $\epsilon \to 0$

Limit of $\lambda^{R,\epsilon}_1(\xi)\mathcal{B}^{R,\epsilon}_1(\psi_0u^\epsilon)$

We substitute the power series expansion of the first Bloch eigenvalue about $\eta = 0$ in $\lambda^{R,\epsilon}_1(\xi)\mathcal{B}^{R,\epsilon}_1(\psi_0u^\epsilon)$ and then pass to the limit $\epsilon \to 0$ in $L^2_{loc}(\mathbb{R}^d_\xi)$-weak by applying Lemma 4.4 to obtain:

$$\frac{1}{2}\frac{\partial^2\lambda^R_1}{\partial\eta_s\partial\eta_t}(0)\xi_s\xi_t\widehat{\psi_0u^*}(\xi). \tag{4.33}$$

Limit of $\mathcal{B}^{R,\epsilon}_1(\psi_0f)$

A simple application of Lemma 4.4 yields the convergence of $\mathcal{B}^{R,\epsilon}_1(\psi_0f)$ to $\widehat{(\psi_0f)}$ in $L^2_{loc}(\mathbb{R}^d_\xi)$-weak.

Limit of $\mathcal{B}^{R,\epsilon}_1g^\epsilon$

The sequence g^ϵ as defined in (4.29) is bounded in $L^2(\mathbb{R}^d)$ and hence has a weakly convergent subsequence with limit $g^* \in L^2(\mathbb{R}^d)$. This sequence is supported in a fixed set K. Also, note that the sequence $\sigma^\epsilon_k(x) := a^\epsilon_{kl}(x)\dfrac{\partial u^\epsilon}{\partial x_l}(x)$ is bounded in $L^2(\Omega)$, hence has a weakly convergent subsequence whose limit is denoted by σ^*_k for $k = 1, 2, \ldots, d$. Extend σ^*_k by zero outside Ω and continue to denote the extension by σ_k^*. Thus, g^* is given by $-\frac{\partial\psi_0}{\partial x_k}\sigma_k^*$. Therefore, by Lemma 4.4, we obtain the following convergence in $L^2_{loc}(\mathbb{R}^d_\xi)$-weak:

$$\chi_{\epsilon^{-1}U_R}(\xi)\mathcal{B}_1^{R,\epsilon}g^\epsilon(\xi) \rightharpoonup -\widehat{\left(\frac{\partial\psi_0}{\partial x_k}(x)\sigma_k^*(x)\right)}(\xi). \tag{4.34}$$

Notice that the limit is independent of R.

LIMIT OF $\mathcal{B}_1^{R,\epsilon}h^{R,\epsilon}$

We have the following weak convergence for $\mathcal{B}_1^{R,\epsilon}h^{R,\epsilon}$ in $L^2_{loc}(\mathbb{R}^d_\xi)$.

$$\lim_{\epsilon\to 0}\chi_{\epsilon^{-1}U_R}(\xi)\mathcal{B}_1^{R,\epsilon}h^{R,\epsilon}(\xi) = -i\xi_k a_{kl}^{R,*}\widehat{\left(\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\right)}(\xi) \tag{4.35}$$

We shall prove this in the following steps.

STEP 1 By the definition of the Bloch transform (4.17) for elements of $H^{-1}(\mathbb{R}^d)$, we have

$$\begin{aligned}\mathcal{B}_1^{R,\epsilon}h^{R,\epsilon}(\xi) = -i\xi_k\int_{\mathbb{R}^d} e^{-ix\cdot\xi}\frac{\partial\psi_0}{\partial x_l}(x)a_{kl}^{R,\epsilon}(x)u^\epsilon(x)\overline{\phi_1^R\left(\frac{x}{\epsilon};\epsilon\xi\right)}\,dx \\ + \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\frac{\partial\psi_0}{\partial x_l}(x)a_{kl}^{R,\epsilon}(x)u^\epsilon(x)\frac{\partial\overline{\phi_1^R}}{\partial x_k}\left(\frac{x}{\epsilon};\epsilon\xi\right)\,dx.\end{aligned} \tag{4.36}$$

STEP 2 The first term on RHS of (4.36) is the Bloch transform of $-i\xi_k\frac{\partial\psi_0}{\partial x_l}(x)a_{kl}^{R,\epsilon}(x)u^\epsilon(x)$ which converges weakly to $-i\xi_k\mathcal{M}(a_{kl}^R)\left(\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\right)$.

STEP 3 Now, we analyze the second term on RHS of (4.36). In order to do this, we use the analyticity of first Bloch eigenfunction with respect to the dual parameter η near 0. We have the following power series expansion in $H^1_\sharp(Y_R)$ for $\phi_1^R(\eta)$ about $\eta = 0$:

$$\phi_1^R(y;\eta) = \phi_1^R(y;0) + \eta_s\frac{\partial\phi_1^R}{\partial\eta_s}(y;0) + \gamma^R(y;\eta). \tag{4.37}$$

We know that $\gamma^R(y;0) = 0$ and $(\partial\gamma^R/\partial\eta_s)(y;0) = 0$, therefore, $\gamma^R(\cdot;\eta) = O(|\eta|^2)$ in $L^\infty(U_R; H^1_\sharp(Y_R))$. We also have $(\partial\gamma^R/\partial y_k)(\cdot;\eta) = O(|\eta|^2)$ in $L^\infty(U_R; L^2_\sharp(Y_R))$. Now,

$$\phi_1^{R,\epsilon}(x;\xi) = \phi_1^R\left(\frac{x}{\epsilon};\epsilon\xi\right) = \phi_1^R\left(\frac{x}{\epsilon};0\right) + \epsilon\xi_s\frac{\partial\phi_1^R}{\partial\eta_s}\left(\frac{x}{\epsilon};0\right) + \gamma^R\left(\frac{x}{\epsilon};\epsilon\xi\right). \tag{4.38}$$

Differentiating the last equation with respect to x_k, we obtain

$$\frac{\partial}{\partial x_k}\phi_1^R\left(\frac{x}{\epsilon};\epsilon\xi\right) = \xi_s\frac{\partial}{\partial x_k}\frac{\partial\phi_1^R}{\partial\eta_s}\left(\frac{x}{\epsilon};0\right) + \epsilon^{-1}\frac{\partial\gamma^R}{\partial y_k}\left(\frac{x}{\epsilon};\epsilon\xi\right). \tag{4.39}$$

For ξ belonging to the set $\{\xi : \epsilon\xi \in U_R \text{ and } |\xi| \le M\}$, we have

$$\frac{\partial \gamma^R}{\partial y_k}(\cdot;\epsilon\xi) = O(|\epsilon\xi|^2) = \epsilon^2 O(|\xi|^2) \le CM^2\epsilon^2. \tag{4.40}$$

As a consequence,

$$\epsilon^{-2}\frac{\partial \gamma^R}{\partial y_k}(x/\epsilon;\epsilon\xi) \in L^\infty_{loc}(\mathbb{R}^d_\xi; L^2_\sharp(\epsilon Y_R)). \tag{4.41}$$

The second term on the RHS of (4.36) is given by

$$\chi_{\epsilon^{-1}U_R}(\xi)\int_K e^{-ix\cdot\xi}\frac{\partial\psi_0}{\partial x_l}(x)a^R_{kl}\left(\frac{x}{\epsilon}\right)u^\epsilon(x)\frac{\partial}{\partial x_k}\left(\overline{\phi^R_1\left(\frac{x}{\epsilon};\epsilon\xi\right)}\right)dx. \tag{4.42}$$

Substituting (4.39) in (4.42), we obtain

$$\chi_{\epsilon^{-1}U_R}(\xi)\int_K e^{-ix\cdot\xi}\frac{\partial\psi_0}{\partial x_l}(x)a^R_{kl}\left(\frac{x}{\epsilon}\right)u^\epsilon(x)\left[\xi_s\frac{\partial}{\partial x_k}\frac{\partial\phi^R_1}{\partial\eta_s}\left(\frac{x}{\epsilon};0\right)+\epsilon^{-1}\frac{\partial\gamma^R}{\partial y_k}\left(\frac{x}{\epsilon};\epsilon\xi\right)\right]dx. \tag{4.43}$$

In the last expression, the term involving γ^R goes to zero as $\epsilon \to 0$ in view of (4.40), whereas the other term has the following limit due to the strong convergence of u^ϵ and weak convergence of $\left(a^R_{kl}(x/\epsilon)\frac{\partial}{\partial x_k}\left(\frac{\partial\phi^R_1}{\partial\eta_s}(x/\epsilon;0)\right)\right)$:

$$\mathcal{M}\left(a^R_{kl}(y)\frac{\partial}{\partial y_k}\left(\frac{\partial\phi^R_1}{\partial\eta_s}(y;0)\right)\right)\xi_s\int_{\mathbb{R}^d} e^{-ix\cdot\xi}\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\,dx. \tag{4.44}$$

STEP 4 By Theorem 4.7 and Remark 4.2, it follows that

$$\mathcal{M}\left(a^R_{kl}(y)\frac{\partial}{\partial y_k}\left(\frac{\partial\phi^R_1}{\partial\eta_s}(y;0)\right)\right) = -i(2\pi)^{-d/2}\mathcal{M}\left(a^R_{kl}(y)\frac{\partial w^{R,s}}{\partial y_k}(y)\right). \tag{4.45}$$

Therefore, we have the following convergence in $L^2_{loc}(\mathbb{R}^d_\xi)$-weak:

$$\begin{aligned}\chi_{\epsilon^{-1}U_R}(\xi)\mathcal{B}^{R,\epsilon}_1 h^{R,\epsilon}(\xi) &\rightharpoonup -i\xi_s\left\{\mathcal{M}(a^R_{kl})+\mathcal{M}\left(a^R_{kl}(y)\frac{\partial w^{R,s}}{\partial y_k}(y)\right)\right\}\widehat{\left(\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\right)}(\xi)\\ &= -i\xi_s a^{R,*}_{kl}\widehat{\left(\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\right)}(\xi)\end{aligned} \tag{4.46}$$

LIMIT OF $\mathcal{B}^{R,\epsilon}_1 l^{R,\epsilon}$

Let $v^{R,\epsilon}_k := \psi_0(x)(a^{R,\epsilon}_{kl}(x) - a^\epsilon_{kl}(x))\frac{\partial u^\epsilon}{\partial x_l}(x)$, then by the definition of the Bloch transform (4.17) for elements of $H^{-1}(\mathbb{R}^d)$, we have

$$\begin{aligned}\mathcal{B}^{R,\epsilon}_1 l^{R,\epsilon}(\xi) = -i\xi_k\int_{\mathbb{R}^d} e^{-ix\cdot\xi}v^{R,\epsilon}_k(x)\overline{\phi^R_1\left(\frac{x}{\epsilon};\epsilon\xi\right)}dx\\ +\int_{\mathbb{R}^d} e^{-ix\cdot\xi}v^{R,\epsilon}_k(x)\frac{\partial\overline{\phi^R_1}}{\partial x_k}\left(\frac{x}{\epsilon};\epsilon\xi\right)dx.\end{aligned} \tag{4.47}$$

The sequence $v_k^{R,\epsilon}$ is bounded in $L^2(\mathbb{R}^d)$, hence converges weakly to a limit $v_k^R \in L^2(\mathbb{R}^d)$. The first term on the RHS of (4.47) is the Bloch transform of $-i\xi_k v_k^{R,\epsilon}$, hence by Lemma 4.4, it converges to $-i\xi_k (v_k^R(x))\widehat{}(\xi)$.

Using equation (4.39), the second term on RHS of (4.47) can be written as

$$\int_{\mathbb{R}^d} e^{-ix\cdot\xi} v_k^{R,\epsilon}(x) \left[\xi_s \frac{\partial}{\partial x_k} \frac{\partial \phi_1^R}{\partial \eta_s}\left(\frac{x}{\epsilon};0\right) + \epsilon^{-1} \frac{\partial \gamma^R}{\partial y_k}\left(\frac{x}{\epsilon};\epsilon\xi\right) \right] dx. \tag{4.48}$$

The second term in the above expression goes to 0 in view of (4.40). The sequence $z_s^{R,\epsilon}(x) := v_k^{R,\epsilon}(x) \frac{\partial}{\partial x_k} \frac{\partial \phi_1^R}{\partial \eta_s}\left(\frac{x}{\epsilon};0\right)$ is bounded in $L^2(\mathbb{R}^d)$. Therefore, it has a weakly convergent subsequence whose limit we shall call z_s^R. The second term on RHS of (4.47) converges to the Fourier transform of $\xi_s z_s^R$.

$$\lim_{\epsilon\to 0} \mathcal{B}_1^{R,\epsilon} l^{R,\epsilon} \to -i\xi_k (v_k^R(x))\widehat{}(\xi) + \xi_s (z_s^R)\widehat{}. \tag{4.49}$$

Finally, passing to the limit in (4.32) as $\epsilon \to 0$ by applying equations (4.33), (4.34), (4.35) and (4.49) we get:

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 \lambda_1^R}{\partial \eta_s \partial \eta_t}(0)\xi_s\xi_t \widehat{\psi_o u^*}(\xi) =& (\psi_0 f)\widehat{}(\xi) - \widehat{\left(\frac{\partial \psi_0}{\partial x_k}(x)\sigma_k^*(x)\right)}(\xi) \\ & - i\xi_s a_{kl}^{R,*} \widehat{\left(\frac{\partial \psi_0}{\partial x_l}(x) u^*(x)\right)}(\xi) - i\xi_k (v_k^R(x))\widehat{}(\xi) + \xi_s (z_s^R)\widehat{}.\end{aligned} \tag{4.50}$$

4.5.3 Limit $R \to \infty$

In the equation (4.50) above, we pass to the limit $R \to \infty$ as follows:

Firstly, observe that

$$||v_k^R||_{L^2(\mathbb{R}^d)} \le \liminf_{\epsilon\to 0} ||v_k^{R,\epsilon}||_{L^2(\mathbb{R}^d)} \le C \max_l ||a_{kl}^{R,\epsilon} - a_{kl}^\epsilon||_{L^\infty(K)} = C \max_l ||a_{kl}^R - a_{kl}||_{L^\infty(\epsilon K)},$$

due to the weak lower semicontinuity of norm. Hence, $v_k^R \to 0$ as $R \to \infty$.

Secondly,

$$||z_s^R||_{L^2(\mathbb{R}^d)} \le \liminf_{\epsilon\to 0} ||z_s^{R,\epsilon}||_{L^2(\mathbb{R}^d)} \le C \max_{k,l} ||a_{kl}^{R,\epsilon} - a_{kl}^\epsilon||_{L^\infty(K)} = C \max_{k,l} ||a_{kl}^R - a_{kl}||_{L^\infty(\epsilon K)},$$

due to the weak lower semicontinuity of norm. Hence, $z_s^R \to 0$ as $R \to \infty$.

As a consequence we obtain the following limit equation in the Fourier space:

$$a_{kl}^*\xi_k\xi_l \widehat{\psi_o u^*}(\xi) = \widehat{\psi_0 f} - \widehat{\left(\frac{\partial \psi_0}{\partial x_k}(x)\sigma_k^*(x)\right)}(\xi) - i\xi_k a_{kl}^* \widehat{\left(\frac{\partial \psi_0}{\partial x_l}(x)u^*(x)\right)}(\xi). \tag{4.51}$$

4.5.4 Proof of the homogenization result

Taking the inverse Fourier transform in the equation (4.51) above, we obtain the following:

$$(\mathcal{A}^{hom}(\psi_0 u^*)(x)) = \psi_0 f - \frac{\partial \psi_0}{\partial x_k}(x)\sigma_k^*(x) - a_{kl}^*\frac{\partial}{\partial x_k}\left(\frac{\partial \psi_0}{\partial x_l}(x)u^*(x)\right), \tag{4.52}$$

where the operator $\mathcal{A}^{hom}$ is defined in (4.27). At the same time, calculating using Leibniz rule, we have:

$$\mathcal{A}^{hom}(\psi_0 u^*)(x) = \psi_0(x)\mathcal{A}^{hom}u^*(x) - a_{kl}^*\frac{\partial}{\partial x_k}\left(\frac{\partial \psi_0}{\partial x_l}(x)u^*(x)\right) - a_{kl}^*\frac{\partial \psi_0}{\partial x_k}(x)\frac{\partial u^*}{\partial x_l}(x) \tag{4.53}$$

Using equations (4.52) and (4.53), we obtain

$$\psi_0(x)\left(\mathcal{A}^{hom}u^* - f\right)(x) = \frac{\partial \psi_0}{\partial x_k}\left[a_{kl}^*\frac{\partial u^*}{\partial x_l}(x) - \sigma_k^*(x)\right]. \tag{4.54}$$

Let ω be a unit vector in $\mathbb{R}^d$, then $\psi_0(x)e^{ix\cdot\omega} \in \mathcal{D}(\Omega)$. On substituting in the above equation, we get, for all $k = 1, 2, \ldots, d$ and for all $\psi_0 \in \mathcal{D}(\Omega)$,

$$\psi_0(x)\left[a_{kl}^*\frac{\partial u^*}{\partial x_l}(x) - \sigma_k^*(x)\right] = 0. \tag{4.55}$$

Let x_0 be an arbitrary point in Ω and let $\psi_0(x)$ be equal to 1 near x_0, then for a small neighborhood of x_0:

$$\text{for } k = 1, 2, \ldots, d, \quad \left[a_{kl}^*\frac{\partial u^*}{\partial x_l}(x) - \sigma_k^*(x)\right] = 0 \tag{4.56}$$

However, $x_0 \in \Omega$ is arbitrary, so that

$$\mathcal{A}^{hom}u^* = f \text{ and } \sigma_k^*(x) = a_{kl}^*\frac{\partial u^*}{\partial x_l}(x). \tag{4.57}$$

Thus,we have obtained the limit equation in the physical space. This finishes the proof of Theorem 4.8.

4.6 Identification of the Homogenized Tensor

In this section, we recall that a_{kl}^* can be identified with the homogenized tensor for the almost periodic operator $\mathcal{A}^\epsilon$ [Koz78, OZ82, JKO94] and that a_{kl}^* does not depend on any subsequence of $a_{kl}^{R,*}$. The study of homogenization of almost periodic media was initiated by Kozlov [Koz78] who also obtained a convergence rate for a subclass of quasiperiodic media. Subsequently, an abstract approach which seeks solutions without derivatives was explained in [OZ82, JKO94] and is described in the next subsection.

4.6.1 Cell Problem for Almost Periodic Media

We begin by introducing the cell problem for almost periodic operator $\mathcal{A}$. Consider the set $S = \{\nabla\phi : \phi \in \mathrm{Trig}(\mathbb{R}^d; \mathbb{R})\}$ as a subset of $(B^2(\mathbb{R}^d))^d$, the Hilbert space of all d-tuples of $B^2(\mathbb{R}^d)$ functions. Let W denote the closure of S in $(B^2(\mathbb{R}^d))^d$. Let $U = (u_1, u_2, \dots, u_d) \in W$ and $V = (v_1, v_2, \dots, v_d) \in W$. On W, define the bilinear form

$$a(U, V) := \mathcal{M}(AU \cdot V). \tag{4.58}$$

Then clearly the bilinear form a is continuous and coercive on W. Let $\xi \in \mathbb{R}^d$. Define a linear form on W by

$$l_\xi(V) := -\mathcal{M}(A\xi \cdot V), \tag{4.59}$$

for $V \in W$. The linear form l_ξ is continuous on W. As a consequence, by Lax-Milgram lemma, the problem

$$a(N^\xi, V) = l_\xi(V),\ \forall V \in W \tag{4.60}$$

has a solution $N^\xi \in W$ and by the classical theory of almost periodic homogenization [OZ82], the homogenized coefficients for $\mathcal{A}^\epsilon$ are given by

$$q^*_{kl} = \mathcal{M}\left(e_k \cdot Ae_l + e_k \cdot AN^{e_l}\right), \tag{4.61}$$

where e_i denotes the unit vector in $\mathbb{R}^d$ with 1 in the i^{th} place and 0 elsewhere.

Since periodic media are also almost periodic, a question arises as to whether the formulation (4.60) is consistent with (4.20).

We restate the two cell problems here in their variational formulations:

The corrector $w^{R,\xi}$ satisfies

$$\mathcal{M}_{Y_R}\left(A^R\nabla w^{R,\xi} \cdot \nabla\phi\right) = -\mathcal{M}_{Y_R}\left(A^R\xi \cdot \nabla\phi\right) \tag{4.62}$$

for all $\phi \in H^1_\sharp(Y_R)$ whereas N^ξ satisfies

$$\mathcal{M}\left(A^R N^{R,\xi} \cdot V\right) = -\mathcal{M}(A^R\xi \cdot V), \tag{4.63}$$

for all $V \in W$.

Lemma 4.9. *Let $w^{R,\xi}$ and $N^{R,\xi}$ satisfy (4.62) and (4.63) respectively, then it holds that*

$$N^{R,\xi} = \nabla w^{R,\xi}.$$

Proof. We will show that $\nabla w^{R,\xi}$ solves the variational formulation (4.63). To see this, it is enough to use test functions $V \in S$. Further, due to linearity, it is enough to use test functions of the form $V = \nabla(e^{iy\cdot\eta})$. Now, observe that if $\eta \in 2\pi R\mathbb{Z}^d$, then $\nabla w^{R,\xi}$ satisfies (4.63) since it reduces to equation (4.62) due to the equality $\mathcal{M}(f) = \mathcal{M}_{Y_R}(f)$ for Y_R-periodic functions f. On the other hand, if $\eta \notin 2\pi R\mathbb{Z}^d$, once again $\nabla w^{R,\xi}$ satisfies (4.63), both sides of which are identically zero, because $\mathcal{M}(f(\cdot)e^{iy\cdot\eta}) = 0$ whenever η is not among the frequencies of f. Hence, in either case, $\nabla w^{R,\xi}$ satisfies equation (4.63). Finally, due to uniqueness,

$$N^{R,\xi} = \nabla w^{R,\xi}.$$

□

Given an almost periodic function f, let $\Lambda(f)$ denote the set of all $\xi \in \mathbb{R}^d$ such that $\mathcal{M}(fe^{-ix\cdot\xi}) \neq 0$. Let $\mathrm{Mod}(f)$ be the $\mathbb{Z}$-module generated by $\Lambda(f)$. The $\mathbb{Z}$-module $\mathrm{Mod}(f)$ shall be referred to as the frequency module of f. In the argument above, we have shown that $\mathrm{Mod}(N^{R,\xi}) \subseteq \mathrm{Mod}(A^R)$. This argument can be readily generalized to a module containment theorem for the correctors. In particular, we may prove that $\mathrm{Mod}(N^{\xi}) \subseteq \mathrm{Mod}(A)$. To paraphrase, the frequencies of the correctors are generated from the frequencies of the coefficients. To this end, we define a closed subspace of the Hilbert space $B^2(\mathbb{R}^d)$ in the following manner. Consider the set of all real trigonometric polynomials whose exponents come from $\mathrm{Mod}(A)$ and call it $\mathrm{Trig}_A(\mathbb{R}^d;\mathbb{R})$. The closure of $\mathrm{Trig}_A(\mathbb{R}^d;\mathbb{R})$ in $B^2(\mathbb{R}^d)$ will be denoted by $B^2_A(\mathbb{R}^d)$. Consider the set $S_A = \{\nabla\phi : \phi \in \mathrm{Trig}_A(\mathbb{R}^d)\}$ as a subset of $(B^2_A(\mathbb{R}^d))^d$, the Hilbert space of all d-tuples of $B^2_A(\mathbb{R}^d)$ functions. Let W_A denote the closure of S_A in $(B^2(\mathbb{R}^d))^d$. To begin with, we prove that the frequencies of a given function $u \in B^2_A(\mathbb{R}^d)$ belong to $\mathrm{Mod}(A)$.

Lemma 4.10. *Let* $u \in B^2_A(\mathbb{R}^d)$. *Let* $\xi \in \mathbb{R}^d$ *such that* $\mathcal{M}(u \cdot e^{ix\cdot\xi}) \neq 0$*, then* $\xi \in \mathrm{Mod}(A)$.

Proof. Since $u \in B^2_A(\mathbb{R}^d)$, we have a sequence of trigonometric polynomials $u_n \in \mathrm{Trig}_A(\mathbb{R}^d)$ such that $\mathcal{M}(|u_n - u|^2) \to 0$. Let $\xi \notin \mathrm{Mod}(A)$, then

$$\begin{aligned}
|\mathcal{M}(u \cdot e^{ix\cdot\xi})| &\leq |\mathcal{M}(u_n \cdot e^{ix\cdot\xi})| + |\mathcal{M}((u_n - u) \cdot e^{ix\cdot\xi})| \\
&= |\mathcal{M}((u_n - u) \cdot e^{ix\cdot\xi})| \\
&\leq \left(\mathcal{M}(|u_n - u|^2)\right)^{1/2},
\end{aligned}$$

which can be made arbitrarily small. Therefore, $\mathcal{M}(u \cdot e^{ix\cdot\xi}) = 0$. □

Now the equation

$$-\operatorname{div}(A(\xi + N)) = 0 \text{ in } \mathbb{R}^d$$

has two variational formulations as below:

Find $N_A^{\xi} \in W_A$ such that

$$\mathcal{M}\Big(AN_A^{\xi} \cdot V\Big) = -\mathcal{M}(A\xi \cdot V), \tag{4.64}$$

for all $V \in W_A$ and find $N^{\xi} \in W$ such that

$$\mathcal{M}\Big(AN^{\xi} \cdot V\Big) = -\mathcal{M}(A\xi \cdot V), \tag{4.65}$$

for all $V \in W$.

Lemma 4.11. *Let* N_A^{ξ} *and* N^{ξ} *satisfy* (4.64) *and* (4.65) *respectively, then it holds that*

$$N^{\xi} = N_A^{\xi}.$$

In particular, $N^{\xi} \in (B_A^2(\mathbb{R}^d))^d$ *and hence* $\mathrm{Mod}(N^{\xi}) \subseteq \mathrm{Mod}(A)$*.*

Proof. We will show that N_A^{ξ} solves the variational formulation (4.65). To see this, it is enough to use test functions $V \in S$. Further, due to linearity, it is enough to use test functions of the form $V = \nabla(e^{iy\cdot\eta})$. Now, observe that if $\eta \in \mathrm{Mod}(A)$, then N_A^{ξ} satisfies (4.65) since it is the same as equation (4.64). On the other hand, if $\eta \notin \mathrm{Mod}(A)$, once again N_A^{ξ} satisfies (4.65), both sides of which are identically zero, because $\mathcal{M}(f(\cdot)e^{iy\cdot\eta}) = 0$ whenever η is not among the frequencies of f. Hence, in either case, N_A^{ξ} satisfies equation (4.65). Finally, due to uniqueness,

$$N^{\xi} = N_A^{\xi}.$$

□

Remark 4.12. By Lemma 4.11, we can conclude that if A is periodic then N^{ξ} is also periodic. Thus, it is possible to conclude Lemma 4.9 from Lemma 4.11. We would also like to point out that Lemma 4.11 is a qualitative version of Theorem 5.6 where the almost periodicity of ∇w^{ξ} is expressed in terms of almost periodicity of A. Module containment results pertaining to a variety of differential equations may be found in [Fin74, AP71].

4.6.2 Convergence of Homogenized Tensors

It was proved by Bourgeat and Piatnitski [BP04, Theorem 1] that approximate homogenized tensors defined in (4.19) using periodic correctors defined in (4.20) converge to the homogenized tensor (4.61) of almost periodic media. They rely on homogenization theorem for almost periodic operators [JKO94, p. 241] and an auxilliary result on convergence of "arbitrary solutions" [JKO94, Theorem 5.2]. We restate this theorem here without proof for which we refer to [BP04].

Theorem 4.13. *(Bourgeat & Piatnistski [BP04, Theorem 1]) Let* $1 \le k, l \le d$ *and let* $a^{R,*}_{kl}$ *and* q^*_{kl} *be defined as in* (4.19) *and* (4.61) *respectively, then* $a^{R,*}_{kl} \to q^*_{kl}$ *as* $R \to \infty$.

In Subsection 4.4.5, we showed that the sequence of homogenized tensors $a^{R,*}_{kl}$ is bounded and hence converges for a subsequence to a limit a^*_{kl}. The theorem of Bourgeat and Piatnitski shows that, in fact, the whole sequence converges to the limit q^*_{kl}. Therefore, $a^*_{kl} = q^*_{kl}$.

Remark 4.14. We can similarly prove that the limit of the fourth-order derivative of the first Bloch eigenvalue at 0 exists. This derivative is called the dispersive tensor [ABV16] or Burnett coefficients [COV06], and is useful in establishing dispersive effective models for long time homogenization of wave propagation in periodic media [DLS14].

4.7 Higher modes do not contribute

The proof of the qualitative homogenization theorem (Theorem 4.8) only requires the first Bloch transform. It is not clear whether the higher Bloch modes make any contribution to the homogenization limit. In this section, we show that they do not. We know that Bloch decomposition is the isomorphism $L^2(\mathbb{R}^d) \cong L^2(Y'; \ell^2(\mathbb{N}))$ which is reflected in the inverse identity (4.14). For simplicity, take $\Omega = \mathbb{R}^d$ and consider the equation $\mathcal{A}^\epsilon u^\epsilon = f$ in $\mathbb{R}^d$ which is equivalent to

$$\mathcal{B}^{R,\epsilon}_m \mathcal{A}^\epsilon u^\epsilon(\xi) = \mathcal{B}^{R,\epsilon}_m f(\xi) \quad \forall m \ge 1, \forall\, \xi \in \epsilon^{-1} Y'_R,$$

which may be further expanded to

$$\mathcal{B}^{R,\epsilon}_m \mathcal{A}^{R,\epsilon} u^\epsilon(\xi) = \mathcal{B}^{R,\epsilon}_m f(\xi) + \left(\mathcal{B}^{R,\epsilon}_m \nabla \cdot (A^\epsilon - A^{R,\epsilon}) \nabla u^\epsilon\right)(\xi) \quad \forall m \ge 1, \forall\, \xi \in \epsilon^{-1} Y'_R,$$

or

$$\lambda^{R,\epsilon}_m(\xi) \mathcal{B}^{R,\epsilon}_m u^\epsilon(\xi) = \mathcal{B}^{R,\epsilon}_m f(\xi) + \left(\mathcal{B}^{R,\epsilon}_m \nabla \cdot (A^\epsilon - A^{R,\epsilon}) \nabla u^\epsilon\right)(\xi) \quad \forall m \ge 1, \forall\, \xi \in \epsilon^{-1} Y'_R. \tag{4.66}$$

We claim that one can neglect all the equations corresponding to $m \geq 2$.

Proposition 4.15. *Let*

$$v^{R,\epsilon}(x) = \int_{\epsilon^{-1}Y'_R} \sum_{m=2}^{\infty} \mathcal{B}_m^{R,\epsilon} u^\epsilon(\xi) \phi_m^{R,\epsilon}(x;\xi) e^{ix\cdot\xi}\, d\xi,$$

then $\|v^{R,\epsilon}\|_{L^2(\mathbb{R}^d)} \leq cR\epsilon$. *Hence, given any sequence,* $\epsilon_k \to 0$*, we can find a sequence* R_k *such that* $v^{R_k,\epsilon_k} \to 0$ *as* $k \to \infty$.

Proof. Due to boundedness of the sequence (u^ϵ) in $H^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \mathcal{A}^{R,\epsilon} u^\epsilon\, \overline{u^\epsilon} \leq C. \tag{4.67}$$

However, by Plancherel Theorem (4.16), we have

$$\int_{\mathbb{R}^d} \mathcal{A}^{R,\epsilon} u^\epsilon\, \overline{u^\epsilon} = \sum_{m=1}^{\infty} \int_{\epsilon^{-1}Y'_R} \left(\mathcal{B}_m^{R,\epsilon} \mathcal{A}^{R,\epsilon} u^\epsilon\right)(\xi)\, \overline{\mathcal{B}_m^{R,\epsilon} u^\epsilon(\xi)}\, d\xi \leq C$$

Using (4.18), we have

$$\sum_{m=1}^{\infty} \int_{\epsilon^{-1}Y'_R} \lambda_m^{R,\epsilon}(\xi) |\mathcal{B}_m^{R,\epsilon} u^\epsilon(\xi)|^2\, d\xi \leq C.$$

Now, by a simple application of Courant-Fischer min-max principle, we can show that

$$\lambda_m^R(\eta) \geq \lambda_2^R(\eta) \geq \lambda_2^R(-\Delta, R) \geq \frac{C}{R^2} > 0 \quad \forall\, m \geq 2 \quad \forall\, \eta \in Y'_R, \tag{4.68}$$

where $\lambda_2^R(-\Delta, R)$ is the second eigenvalue of Laplacian on Y_R with Neumann boundary condition on ∂Y_R. The bound quoted is standard for the Neumann Laplacian on a rectangle but it may also be understood as an instance of the fundamental gap inequality for Neumann Laplacian on convex domains [PW60]. We also know that $\lambda_m^{R,\epsilon}(\xi) = \epsilon^{-2}\lambda_m^{R,\epsilon}$, therefore, combining these two facts, we obtain

$$\sum_{m=2}^{\infty} \int_{\epsilon^{-1}Y'_R} |\mathcal{B}_m^{R,\epsilon} u^\epsilon(\xi)|^2\, d\xi \leq CR^2\epsilon^2.$$

Now, given any sequence $\epsilon_k \to 0$, we can choose a sequence $R_k \to \infty$ such that $R_k^2 < \frac{1}{\epsilon}$, then along this sequence

$$\sum_{m=2}^{\infty} \int_{\epsilon^{-1}Y'_R} |\mathcal{B}_m^{R,\epsilon} u^\epsilon(\xi)|^2\, d\xi \leq C\epsilon.$$

By Parseval's identity, the left side is equal to $\|v^{R,\epsilon}\|^2_{L^2(\mathbb{R}^d)}$. This completes the proof of this Proposition. ∎

Remark 4.16. The product "$R\epsilon$" is the resonance error [Glo11], [GH16] due to the approximation. The above discussion explains the relation between higher modes of the Bloch spectrum and the resonance error. In particular, the periodic approximation serves to separate the lowest Bloch mode from the rest of the spectrum and the limit $R \to \infty$ represents the loss of simplicity and hence analyticity of the lowest Bloch eigenvalue near zero.

4.8 Comments

In [Tar09] , Tartar writes *"Although Bloch waves require a periodic medium in order to be defined, it is clear that physicists use some results from the theory in situations which are not exactly periodic, because of defects for example. It should be useful to understand if there is a natural topology for measuring how far a material is from a periodic medium, so that some results from the theory of Bloch waves still apply."*

It appears that a perturbative method such as ours can model such a distance from periodicity, for example, defects, almost periodicity, transmission problems. The overarching theme of this thesis has been replacement of the original partial differential operator with approximations that have properties desirable for a Bloch wave analysis. We hope to apply such methods to a variety of problems.

On the other hand, a non-commutative notion [BT81] of Bloch decomposition exists in the literature. Although this is couched in the difficult language of C^* algebras and non-commutative integration theory, the apparatus is exact rather than an approximation. We would like to explore the relation between our work and these tools. In particular, the shrinking of the Brillouin zone for the approximations seems to be related to the notion of tiny spaces in the work of Bellisard. One also wonders whether these tools have any computational advantages over the existing method.

In this chapter, we rely on the theorem of Bourgeat and Piatnitski [BP04] for the convergence of approximate homogenized tensors associated with the periodizations. We would like to prove this result independently preferably using methods from spectral theory. However, in homogenization, one often encounters sequences with multiple limit points. Also, the almost periodic operator is not necessarily periodic therefore Bloch wave theory is not expected to be enough for such an analysis.

Chapter 5

Approximation of effective coefficients of almost periodic media

In homogenization of almost periodic media, the cell problem is posed on $\mathbb{R}^d$ and the homogenized coefficients are mean values of almost periodic functions. In practice, the homogenized coefficients are computed using approximate cell problems posed on cubes of increasing size with different boundary conditions. In this paper, we prove that approximation of homogenized tensor using cell problem with Dirichlet boundary conditions converge to the homogenized tensor of almost periodic media. We also provide an estimate for the rate of convergence under a suitable decay hypothesis on a modulus of almost periodicity defined in [ACS14]. The theoretical results are supplemented with numerical study.

5.1 Introduction

Although we are not able to obtain a rate of convergence for the approximate homogenized tensors corresponding to periodization, in this chapter, we obtain a rate of convergence for Dirichlet approximations of homogenized tensors for a class of almost periodic media. To this end, we use ideas from [BP04], [She15] and [Glo11]. Bourgeat and Piatnitski [BP04] prove a convergence rate for approximate homogenized coefficients for stochastic media under strong mixing conditions. The stochastic process generated by almost periodic media is strictly ergodic and not mixing [Sim82]. Therefore the result of Bourgeat and Piatnitski does not apply to them. This necessitates a quantification of almost periodicity. One such quantification is proposed in [ACS14]. We would also like to point out that an important assumption for obtaining rate of convergence for periodic approximations of stochastic media [Fis19] is that coefficients restricted to cubes should follow the same statistics as the coefficient field on $\mathbb{R}^d$. A similar criterion for almost periodic media is open.

5.2 Rate of convergence for Dirichlet approximations

Let a^*_{kl} denote the $(k,l)^{th}$ entry of the homogenized tensor for the almost periodic operator. We shall mostly write this as $e_k \cdot A^* e_l$. Similarly, the homogenized tensor associated to the periodization A^R will be denoted by $A^{R,*}$. In Section 4.6, we observed that $A^{R,*} \to A^*$. It would have been ideal to obtain rate of convergence estimates for the error $|A^* - A^{R,*}|$, however we have been unable to do so. In lieu of this, we provide rate of convergence for Dirichlet approximations to the homogenized tensor in this section. We also carry out a numerical study with some benchmark examples. Dirichlet approximations to the homogenized coefficients are obtained by constructing the approximate correctors with Dirichlet boundary conditions as opposed to the earlier proposed periodic ones. In what follows, we shall also discuss the difficulties in proving the rate of convergence for $A^{R,*}$ to A^*.

5.2.1 Volume Averaging Method

In engineering, the Volume averaging method [BQW88] is employed to determine effective behavior of heterogeneous media by using averages of physical quantities, such as energy, on a large volume of the domain under consideration, called a Representative Elementary Volume [Whi13]. A comparison between the mathematical theory of homogenization and volume averaging is carried out in [DBB+13]. In a well-known paper of Bourgeat and Piatnitski [BP04], the volume averaging technique has been employed to obtain approximations to homogenized tensor for stochastic media.

The homogenization of stochastic as well as almost periodic media has two major difficulties - the cell problem is posed on $\mathbb{R}^d$ and the loss of differential structure, i.e., the correctors do not appear as derivatives in the cell problem (4.60) in almost periodic and stochastic homogenization. The differential structure is important as it is responsible for the compensated compactness of the oscillating test functions in homogenization [CK97]. As a compromise, many authors such as Kozlov [Koz79], Yurinski [Yur86] have introduced cell problems with a penalization (or regularization) term to recover the differential structure. However, these problems are still posed on $\mathbb{R}^d$. The homogenized tensor appears as a mean value on $\mathbb{R}^d$ which makes the computation of homogenized tensor impossible. Hence, volume averages on large cubes provide a suitable proxy for the homogenized tensor.

Like stochastic media, almost periodic media exhibits *long range order*. Stochastic media is quantified in terms of mixing coefficients. In contrast, the process generated by almost periodic media is not mixing, although it is ergodic [Sim82]. In some sense, almost periodic functions fall half way between periodic and random media. Therefore, a quantification specific to almost periodicity is required in order to obtain quantitative results in homogenization theory. A modulus $\rho(A)$ of almost periodicity is defined in [ACS14], which has been employed by Shen [She15] to extend the compactness methods in [AL87] to almost periodic homogenization. Shen also proves that the small divisors condition of Kozlov [Koz78] implies a decay hypothesis on $\rho(A)$. Kozlov was the first to prove a rate of convergence estimate in homogenization of almost periodic media satisfying the small divisors condition. Thereafter, quantitative homogenization of almost periodic operators has seen a resurgence in the works of Armstrong, Shen and coauthors [ACS14, She15, SZ18, AS16, AGK16]. In particular, they extend the regularity theory in homogenization using compactness methods, which was pioneered by Avellaneda and Lin [AL87].

In the next subsection, we shall introduce the Dirichlet approximations, for which we prove a rate of convergence under a suitable decay hypothesis on the modulus of almost periodicity of the almost periodic media. Previously, rate of convergence for approximations to homogenized tensors of almost periodic media have been considered in [GH16] under the small divisors condition of Kozlov.

The contents of this chapter form a section of the preprint [2].

Remark 5.1. As pointed out in the introduction, to obtain rate of convergence for approximate homogenized tensors corresponding to periodization of stochastic media, one requires that A^R follows the same probability distribution as A [Fis19]. An equivalent question for almost periodic media would be whether the periodization A^R has the "same" almost periodicity as A.

5.2.2 Dirichlet Approximations of Cell Problem

The cell problem for almost periodic media (4.60) is posed in $\mathbb{R}^d$. The following is its Dirichlet approximation, which is the truncation of (4.60) on a cube $Y_R = [-R\pi, R\pi)^d$ of side length $2\pi R$. Let $H_0^1(Y_R)$ denote the space of all $L^2(Y_R)$ functions whose weak derivatives are also in $L^2(Y_R)$ and whose trace on Y_R is zero.

Given $\xi \in \mathbb{R}^d$, find $w^{R,D,\xi} \in H_0^1(Y_R)$ such that

$$-\nabla \cdot A(\xi + \nabla w^{R,D,\xi}) = 0. \tag{5.1}$$

Then Dirichlet approximation $A^{R,D,*} = \left(a_{kl}^{R,D,*}\right)$ to the homogenized tensor is given by

$$a_{kl}^{R,D,*} = \mathcal{M}_{Y_R}\left(a_{kl} + \sum_{j=1}^{d} a_{kj}\frac{\partial w^{R,D,e_l}}{\partial y_j}\right). \tag{5.2}$$

5.2.3 Convergence Result

The Dirichlet approximations for the homogenized tensor converge to the homogenized tensor of almost periodic operators. This is the content of the next theorem whose proof is omitted since it may be found in [BP04, Theorem 2].

Theorem 5.2. *(Bourgeat & Piatnistski [BP04, Theorem 2]) Let* $A^{R,D,*}$ *be defined as in* (5.2) *and let* A^* *be defined as in* (4.61), *then* $A^{R,D,*} \to A^*$ *as* $R \to \infty$.

5.2.4 Rate of convergence estimates

In this subsection, we will estimate the error $|A^* - A^{R,D,*}|$ using the strategy of Bourgeat and Piatnitski [BP04]. Their techniques were refined and improved by Gloria and his coauthors [Glo11], [GH16], [GO17]. We shall follow the ideas of these authors to establish convergence rate for $A^{R,D,*}$ in terms of the following quantification of almost periodicity as introduced in [ACS14]. For a fixed $L > 0$ and a matrix A with entries in $L^\infty(\mathbb{R}^d)$, define the following modulus of almost periodicity:

$$\rho(A, L) := \sup_{y \in \mathbb{R}^d} \inf_{|z| \le L} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)}. \tag{5.3}$$

It follows that A is uniformly almost periodic if and only if $\rho(A, L) \to 0$ as $L \to \infty$. In particular, for periodic functions, the modulus becomes zero for large L. We are now ready to state the theorem on the rate of convergence.

Theorem 5.3. *If* $A \in AP(\mathbb{R}^d)$ *is such that, for each* $L > 0$, $\rho(A, L)$ *satisfies* $\rho(A, L) \lesssim 1/L^{\tau}$ *for some* $\tau > 0$, *then, there exists a* $\beta \in (0, 1)$ *such that*

$$|A^* - A^{R,D,*}| \lesssim \frac{1}{R^{\beta}}, \tag{5.4}$$

where A^* *and* $A^{R,D,*}$ *are defined in* (4.61) *and* (5.2) *respectively.* □

5.2.5 Strategy of Proof

The proof of Theorem 5.3 will be done in four steps. We have already seen two cell problems corresponding to the almost periodic media and its Dirichlet approximation, viz., (4.60) and (5.1). We shall require two more cell problems, corresponding to regularization of (4.60) and (5.1). For the sake of convenience, we list all the requisite cell problems below. For $\xi \in \mathbb{R}^d$ and $T > 0$:

(D) Find $w^{R,D,\xi} \in H_0^1(Y_R)$ such that

$$-\nabla \cdot (A(\xi + \nabla w^{R,D,\xi})) = 0. \tag{5.5}$$

(DT) Find $w_T^{R,D,\xi} \in H_0^1(Y_R)$ such that

$$-\nabla \cdot (A(\xi + \nabla w_T^{R,D,\xi})) + T^{-1} w_T^{R,D,\xi} = 0. \tag{5.6}$$

(AP) Find $N^{\xi} \in (B^2(\mathbb{R}^d))^d$ such that

$$\mathcal{M}\left(AN^{\xi} \cdot v\right) = -\mathcal{M}(A\xi \cdot v) \tag{5.7}$$

for all $v \in \{\nabla\phi : \phi \in \text{Trig}(\mathbb{R}^d)\}$.

(APT) Find $w_T^{\xi} \in H_{loc}^1(\mathbb{R}^d)$ such that

$$-\nabla \cdot (A(\xi + \nabla w_T^{\xi})) + T^{-1} w_T^{\xi} = 0. \tag{5.8}$$

The homogenized tensor A^* is defined as

$$\xi \cdot A^* \xi = \mathcal{M}\Big((\xi + N^{\xi}) \cdot A(\xi + N^{\xi})\Big).$$

Define A_T^* as

$$\xi \cdot A_T^* \xi = \mathcal{M}\Big((\xi + \nabla w_T^{\xi}) \cdot A(\xi + \nabla w_T^{\xi})\Big). \tag{5.9}$$

Also, define the truncated average $\overline{A}_{T,R}$ as

$$\xi \cdot \overline{A}_{T,R} \xi = \frac{1}{|Y_R|} \int_{Y_R} \Big((\xi + \nabla w_T^{\xi}) \cdot A(\xi + \nabla w_T^{\xi})\Big)\, dy,$$

and define the Dirichlet approximation $A^{R,D,*}$ to A^* as

$$\xi \cdot A^{R,D,*}\xi = \frac{1}{|Y_R|}\int_{Y_R} \left((\xi + \nabla w^{R,D,\xi}) \cdot A(\xi + \nabla w^{R,D,\xi})\right)\, dy.$$

The homogenized tensor corresponding to the regularized Dirichlet cell problem (5.6) is

$$\xi \cdot A_T^{R,D,*}\xi = \frac{1}{|Y_R|}\int_{Y_R} \left((\xi + \nabla w_T^{R,D,\xi}) \cdot A(\xi + \nabla w_T^{R,D,\xi})\right)\, dy. \tag{5.10}$$

With the notation in place, we can proceed with the strategy for obtaining the rate of convergence estimates. This is essentially the same as the one employed by Bourgeat and Piatnitski [BP04] to obtain estimates for Dirichlet approximations of homogenized tensor for random ergodic media. We shall write

$$|A^* - A^{R,D,*}| \leq |A^* - A_T^*| + |A_T^* - \overline{A}_{T,R}| + |\overline{A}_{T,R} - A_T^{R,D,*}| + |A_T^{R,D,*} - A^{R,D,*}| \tag{5.11}$$

In the above inequality, the first and last terms on RHS are estimated in terms of the rate of convergence of regularized correctors to the exact correctors as $T \to \infty$. The proof of this estimate for the first term is available in Shen [She15]. For the proof of estimate for the last term, we adapt the argument in Bourgeat and Piatnitski [BP04].

The second term corresponds to rate of convergence in mean ergodic theorems. This estimate is available for periodic and quasiperiodic functions and is of order $1/R$. In Blanc and Le Bris [BLB10] and Gloria [Glo11], a different truncated approximation is proposed, through the use of filters; either as a weight in the cell problem or as post-processing. Such approximations have faster rates of convergence. However, we shall write this rate of convergence in terms of $\rho(A, L)$ following [SZ18].

The third term on RHS corresponds to a boundary term which is controlled by the Green's function decay of the regularized operator $T^{-1} - \nabla \cdot (A\nabla)$ in $\mathbb{R}^d$. The proof is essentially due to Bourgeat and Piatnitski [BP04] for stochastic media but has lately been refined by Gloria [Glo11] for general media (also see [GO17]).

In the next subsections, we shall prove the four convergence rates.

5.2.6 Rate of convergence of regularized correctors

We will begin by establishing the existence of the regularized correctors as defined in (5.8). This can be done in two ways. One is by following the derivation theory of Besicovitch spaces as presented in Casado-Díaz and Gayte [CDG02].

The other method is to build solutions in $H^1_{loc}(\mathbb{R}^d)$ directly by approximations on disks [PY89], [She15]. The second method is more general as it does not require the assumption of almost periodicity on the coefficients. However, the existence of a derivation theory on Besicovitch spaces makes it easier to obtain a priori estimates.

For $p \in (1,\infty)$, $B^p(\mathbb{R}^d)$ is the closure of trigonometric polynomials in the semi-norm $\mathcal{M}(|\cdot|^p))^{1/p}$. Let D^∞ be the space

$$D^\infty := \{\, \phi \in C^\infty(\mathbb{R}^d) \;:\; D^\alpha\phi \in B^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \text{ for all multiindices } \alpha \}, \tag{5.12}$$

which is analogous to the space of test functions for defining weak derivatives in the theory of distributions. Next, given a function $u \in B^1(\mathbb{R}^d)$, define its mean derivative $\partial_j u$ as a linear map on D^∞ given by $\partial_j u(\phi) := -\mathcal{M}\left(u\frac{\partial\phi}{\partial x_j}\right)$. This definition is well defined in the sense that if u_1 and u_2 are two functions in $B^1(\mathbb{R}^d)$ such that $\mathcal{M}(|u_1 - u_2|) = 0$, then they define the same mean derivative. Moreover, if the distributional derivative of a function $u \in B^1(\mathbb{R}^d)$ is also in $B^1(\mathbb{R}^d)$, then it agrees with the mean derivative of u. The following definition of the Besicovitch analogue of Sobolev spaces is presented in [CDG02]:

$$B^{1,p}(\mathbb{R}^d) := \{\, u \in B^p(\mathbb{R}^d) \;:\; \exists\, u_j \in B^p(\mathbb{R}^d) \text{ such that } \partial_j u(\phi) = \mathcal{M}(u_j\phi),\ 1 \le j \le d \}. \tag{5.13}$$

This space admits the semi-norm

$$|u|_{\mathcal{M}} = \mathcal{M}(|u|) + \mathcal{M}(|\nabla u|).$$

It can be made into a Banach space by identifying those elements whose difference has zero semi-norm. We shall continue to denote the associated Banach space as $B^{1,p}(\mathbb{R}^d)$. Further, every representative u is an element of $W^{1,p}_{loc}(\mathbb{R}^d)$ with the property that any two representatives u_1 and u_2 satisfy $|u_1 - u_2|_{\mathcal{M}} = 0$.

Theorem 5.4. *Let the matrix* A *satisfy (D1), (D2), (D3). Then equation* (5.8) *has a unique solution* $w_T^\xi \in B^{1,2}(\mathbb{R}^d)$, *and*

$$T^{-1}\mathcal{M}(|w_T^\xi|^2) + \mathcal{M}(|\nabla w_T^\xi|^2) \lesssim 1. \tag{5.14}$$

Proof. The space $B^{1,2}(\mathbb{R}^d)$ is a Hilbert space. Define the bilinear form

$$a(w,v) := \mathcal{M}(A\nabla w \nabla v + T^{-1}wv),$$

which is elliptic due to coercivity of A. Also, define the linear form

$$l(v) := -\mathcal{M}(A\xi\cdot\nabla v)$$

for $v \in B^{1,2}(\mathbb{R}^d)$. The equation (5.8) is said to have a solution in $B^{1,2}(\mathbb{R}^d)$ if there exists $w_T^\xi \in B^{1,2}(\mathbb{R}^d)$ such that $a(w_T^\xi, v) = l(v)$ for all $v \in D^\infty$. The existence and uniqueness of such a solution is guaranteed by an application of Lax-Milgram lemma. Each representative of $w_T^\xi \in B^{1,2}(\mathbb{R}^d)$ is an element of $H^1_{loc}(\mathbb{R}^d)$. The estimate (5.14) is obtained from the weak formulation by choosing $v = w_T^\xi$ followed by an application of Young's inequality. □

The convergence rate for the first term in (5.11) is available in Shen [She15] in terms of the function $\rho(A, \cdot)$.

Theorem 5.5 (Shen [She15], Remark 6.7). *Let $A \in AP(\mathbb{R}^d)$ be such that, for each $L > 0$, $\rho(A, L)$ satisfies $\rho(A, L) \lesssim 1/L^\tau$ for some $\tau > 0$. Then, for any ω such that $0 < \omega < 1$,*

$$|A^* - A_T^*| \leq CT^{-\frac{\tau}{2(\tau+1)}+\omega}, \tag{5.15}$$

where the constant is independent of T but depends on ω and A^* and A_T^* are defined in (4.61) and (5.9) respectively.

5.2.7 Rate of convergence of truncated homogenized tensor

In proving the convergence of truncated averages $\overline{A}_{T,R}$ to A_T^*, we need to show that the almost periodicity of the correctors w_T^ξ can be quantified in terms of the almost periodicity of A. This is the content of the following theorem from Shen [She15].

Theorem 5.6. *(Shen [She15], Lemma 5.3) For $y, z \in \mathbb{R}^d$, the regularized corrector w_T^ξ satisfies*

$$\left(\fint_{Y_R} |\nabla w_T^\xi(t+y) - \nabla w_T^\xi(t+z)|^2 \, dt \right)^{1/2} \leq C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)}, \tag{5.16}$$

where C is independent of R, y, and z.

Further, Shen and Zhuge [SZ18] have quantified the convergence of truncated averages in terms of almost periodicity of the integrands in the following theorem.

Theorem 5.7. *(Shen & Zhuge [SZ18]) For $1 < p < \infty$ let $u \in B^p(\mathbb{R}^d)$ and for $p = \infty$ let $u \in AP(\mathbb{R}^d)$. Then for any $0 < L \leq R < \infty$,*

$$\left| \fint_{Y_R} u \, dy - \mathcal{M}(u) \right| \lesssim \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \fint_{Y_R} |u(t+y) - u(t+z)| \, dt + \left(\frac{L}{R}\right)^{1/p'} \begin{cases} \|u\|_{B^p} & \text{if } p < \infty \\ \|u\|_{L^\infty} & \text{if } p = \infty \end{cases} \tag{5.17}$$

As a consequence of the two theorems stated above, we can prove the rate of convergence estimate $|\overline{A}_{T,R} - A_T^*|$.

Theorem 5.8. *Let $\rho(A, L)$ satisfy $\rho(A, L) \lesssim 1/L^\tau$ for some $\tau > 0$. Then for any $0 < L \le R < \infty$,*

$$|\overline{A}_{T,R} - A_T^*| \lesssim \frac{1}{L^\tau} + \left(\frac{L}{R}\right)^{1/2}. \tag{5.18}$$

Proof. We shall apply Theorem 5.7 to the functions $u_1 = e_k \cdot A e_l$ and $u_2 = e_k \cdot A \nabla w_T^{e_l}$. For u_1, we may choose $p = \infty$ to obtain the following estimate.

$$|\mathcal{M}(A) - \mathcal{M}_{Y_R}(A)| \lesssim \rho(A, L) + \frac{L}{R} \lesssim \frac{1}{L^\tau} + \frac{L}{R}. \tag{5.19}$$

For u_2, we may choose $p = 2$. By Theorem 5.7, we have

$$|\mathcal{M}(A\nabla w_T^{e_l}) - \mathcal{M}_{Y_R}(A\nabla w_T^{e_l})| \lesssim \sup_{y\in\mathbb{R}^d} \inf_{|z|\le L} \fint_{Y_R} |(A\nabla w_T^{e_l})(t+y) - (A\nabla w_T^{e_l})(t+z)|\, dt + \left(\frac{L}{R}\right)^{1/2} \|u\|_{B^2} \tag{5.20}$$

Through an application of Theorem 5.6, we note that

$$\fint_{Y_R} |(A\nabla w_T^{e_l})(t+y) - (A\nabla w_T^{e_l})(t+z)|\, dt \le C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)}. \tag{5.21}$$

Combining (5.20) and (5.21), we get

$$\begin{aligned} |\mathcal{M}(A\nabla w_T^{e_l}) - \mathcal{M}_{Y_R}(A\nabla w_T^{e_l})| &\lesssim \sup_{y\in\mathbb{R}^d} \inf_{|z|\le L} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + \left(\frac{L}{R}\right)^{1/2} \|u\|_{B^2} \\ &\lesssim \frac{1}{L^\tau} + \left(\frac{L}{R}\right)^{1/2}. \end{aligned} \tag{5.22}$$

Combining (5.19) and (5.22), we get (5.18). □

5.2.8 Rate of convergence of boundary term

Now, we shall prove estimate on the boundary term, viz., $|\overline{A}_{T,R} - A_T^{R,D,*}|$. The proof is essentially the same as in [BP04], although the Green's function estimates are borrowed from [GO17]. We begin by recalling the existence of Green's function associated with the operator $T^{-1} - \nabla \cdot (A\nabla)$ and its pointwise bounds.

Theorem 5.9. *(Gloria & Otto [GO17]) Let* A *be a coercive matrix with measurable and bounded entries, and let* $T > 0$*. Then for all* $y \in \mathbb{R}^d$*, there is a function* $G_T(\cdot, y)$ *which is the unique solution in* $W^{1,1}(\mathbb{R}^d)$ *of the equation*

$$T^{-1}G_T(x,y) - \nabla_x \cdot (A\nabla_x G_T(x,y)) = \delta(x-y), \tag{5.23}$$

in the sense of distributions. The function $G_T(\cdot, y)$ *is continuous on* $\mathbb{R}^d \setminus \{y\}$*. Furthermore, the Green's function satisfies the following pointwise bounds:*

$$0 \le G_T(x,y) \lesssim \exp\left(-c\frac{|x-y|}{\sqrt{T}}\right)\begin{cases} \ln\left(2+\frac{\sqrt{T}}{|x-y|}\right) & \text{if } d=2 \\ |x-y|^{2-d}, & \text{if } d>2 \end{cases}. \tag{5.24}$$

Theorem 5.10. *Let* $0 < \delta < 1$, $|\overline{A}_{T,R} - A_T^{R,D,*}| \lesssim R^{(\delta-1)/2} + \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right)\begin{cases} R^d & d>2 \\ R^3 & d=2. \end{cases}$

Proof. Let $R \ge R_0 > 0$ and $\delta \ge \delta_0 > 0$. The proof will be done in three steps: first to obtain an interior estimate in Y_{R-R^δ}, second to obtain an estimate for the boundary layer $Y_R \setminus Y_{R-R^\delta}$ and the final step to obtain the required convergence rate.

Step 1. w_T^ξ satisfies the following equation in $\mathbb{R}^d$:

$$-\nabla \cdot (A(\xi + \nabla w_T^\xi)) + T^{-1}w_T^\xi = 0.$$

$w_T^{R,D,\xi}$ satisfies the following equation in Y_R:

$$-\nabla \cdot (A(\xi + \nabla w_T^{R,D,\xi})) + T^{-1}w_T^{R,D,\xi} = 0.$$

Hence, their difference satisfies

$$T^{-1}(w_T^\xi - w_T^{R,D,\xi}) - \nabla \cdot (A\nabla(w_T^\xi - w_T^{R,D,\xi})) = 0 \text{ in } Y_R$$

in the sense of distributions. Set $\phi_1 = \chi w_T^\xi$, where $\chi \in C^\infty(\overline{Y_R}; \mathbb{R}^+)$, so that $\chi|_{\partial Y_R} = 1$, $\chi|_{Y_{R-R^\delta/2}} = 0$ and $|\nabla\chi| \lesssim 1/R$.

Therefore, by the bounds (5.14), $\|\phi_1\|^2_{L^2(Y_R)} \lesssim R^d T$ and $\|\nabla\phi_1\|^2_{L^2(Y_R)} \lesssim R^d$ for $R \lesssim T \lesssim R^2$.

Now, define $\phi_2 = w_T^\xi - w_T^{R,D,\xi} - \phi_1$, then ϕ_2 satisfies the following equation:

$$T^{-1}\phi_2 - \nabla \cdot A\nabla\phi_2 = -T^{-1}\phi_1 + \nabla \cdot A\nabla\phi_1 \text{ in } Y_R$$
$$\phi_2 = 0 \text{ on } \partial Y_R.$$

Hence, we may write

$$\phi_2(x) = -\int_{Y_R} T^{-1}\phi_1(y)G_{T,R}(x,y) + A(y)\nabla\phi_1(y)\cdot\nabla G_{T,R}(x,y)\, dy, \tag{5.25}$$

where $G_{T,R}$ is the Green's function for the operator $T^{-1} - \nabla \cdot A\nabla$ on Y_R with zero Dirichlet boundary conditions, i.e.,

$$\begin{aligned} T^{-1}G_{T,R}(x,y) - \nabla_x \cdot (A\nabla_x G_{T,R}(x,y)) &= \delta(x-y) && \text{in } Y_R \\ G_{T,R}(x,y) &= 0 && \text{on } \partial Y_R \end{aligned} \tag{5.26}$$

in the sense of distributions. Therefore,

$$\begin{aligned} |\phi_2(x)| \leq \|\phi_1\|_{L^2(Y_R)} &\left(T^{-1} \int_{Y_R \setminus Y_{R-R^\delta/2}} G_{T,R}^2(x,y)\ dy \right)^{1/2} \\ &+ \|A\|_{L^\infty} \|\nabla\phi_1\|_{L^2(Y_R)} \left(\int_{Y_R \setminus Y_{R-R^\delta/2}} |\nabla G_{T,R}(x,y)|^2\ dy \right)^{1/2}. \end{aligned}$$

In the above inequality, the second term will be handled by using Caccioppoli's inequality. In particular, let us multiply the equation (5.26) for Green's function $G_{T,R}$ by $\eta^2 G_{T,R}$ (where η is to be chosen later) and integrate by parts to obtain:

$$\begin{aligned} 0 &= T^{-1} \int_{Y_R} \eta^2(y) G_{T,R}^2(x,y)\ dy + \int_{Y_R} A(y)\nabla(\eta^2(y)G_{T,R}(x,y)) \cdot \nabla G_{T,R}(x,y)\ dy \\ &= T^{-1} \int_{Y_R} \eta^2(y) G_{T,R}^2(x,y)\ dy + \int_{Y_R} A(y)\nabla(\eta(y)G_{T,R}(x,y)) \cdot \nabla(\eta(y)G_{T,R}(x,y))\ dy \\ &\quad - \int_{Y_R} G_{T,R}^2(x,y) A(y)\nabla\eta(y) \cdot \nabla\eta(y), \end{aligned}$$

given that η is zero in some neighborhood of 0. From the last equality, we obtain

$$\int_{Y_R} |\nabla(\eta\ G_{T,R})|^2\ dy \lesssim \int_{Y_R} G_{T,R}^2 |\nabla\eta|^2\ dy.$$

Choose the function $\eta \in C^\infty(Y_R, \mathbb{R}_+)$, such that

$$\begin{cases} \eta = 0 & \text{in } Y_{R-3R^\delta/4}, \\ \eta = 1 & \text{in } Y_R \setminus Y_{R-R^\delta/2}, \\ |\nabla\eta| \lesssim 1/R & , \end{cases} \tag{5.27}$$

then the preceding inequality becomes

$$\int_{Y_R \setminus Y_{R-R^\delta/2}} |\nabla G_{T,R}|^2\ dy \lesssim \frac{1}{R^2} \int_{Y_R \setminus Y_{R-3R^\delta/4}} G_{T,R}^2\ dy.$$

Therefore, for all $x \in Y_R$, we have

$$\begin{aligned} |\phi_2(x)| \lesssim \|\phi_1\|_{L^2(Y_R)} &\left(T^{-1} \int_{Y_R \setminus Y_{R-R^\delta/2}} G_{T,R}^2(x,y)\ dy \right)^{1/2} \\ &+ \|\nabla\phi_1\|_{L^2(Y_R)} \left(\int_{Y_R \setminus Y_{R-3R^\delta/4}} R^{-2} G_{T,R}^2(x,y)\ dy \right)^{1/2}. \end{aligned}$$

For $x \in Y_{R-5R^\delta/6}$, and $y \in Y_R \setminus Y_{R-R^\delta/2}$, we have $\|x-y\|_\infty \geq \|y\|_\infty - \|x\|_\infty \geq R - R^\delta/2 - R + 5R^\delta/6 = R^\delta/3$. Therefore, $|x-y| \gtrsim R^\delta$ Further, note that due to maximum principle, $0 \leq G_{T,R} \leq G_T$. Hence, on using the pointwise estimate for G_T (Theorem 5.9), the above inequality becomes for $d > 2$ and for $x \in Y_{R-5R^\delta/6}$:

$$
\begin{aligned}
|\phi_2(x)| &\lesssim \|\phi_1\|_{L^2(Y_R)} \left(T^{-1} \int_{Y_R \setminus Y_{R-R^\delta/2}} G_T^2(x,y)\, dy \right)^{1/2} \\
&\quad + \|\nabla\phi_1\|_{L^2(Y_R)} \left(\int_{Y_R \setminus Y_{R-3R^\delta/4}} R^{-2} G_T^2(x,y)\, dy \right)^{1/2} \\
&\lesssim R^{d/2} R^{2\delta - d\delta} \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right) R^{d/2} + R^{d/2} R^{-1} R^{2\delta - d\delta} \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right) R^{d/2} \\
&\lesssim R^{d-2\delta+d\delta} \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right).
\end{aligned}
$$

in the regime $R \lesssim T \lesssim R^2$. Similar calculations provide the estimate for $d = 2$.

Hence,

$$
\left(\int_{Y_{R-5R^\delta/6}} |\phi_2(x)|^2\, dx \right)^{1/2} \lesssim R^{d+2\delta-d\delta} R^{d/2} \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right).
$$

Finally, by an application of Caccioppoli's inequality, we have

$$
\left(\int_{Y_{R-R^\delta}} |\nabla\phi_2(x)|^2\, dx \right)^{1/2} \lesssim R^{d/2} R^{d+3\delta-d\delta} \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right).
$$

Therefore,

$$
\left(\int_{Y_{R-R^\delta}} |\nabla(w_T^\xi(x) - w_T^{R,D,\xi}(x))|^2\, dx \right)^{1/2} \lesssim R^{d/2} R^{d+3\delta-d\delta} \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right).
$$

Thus,

$$
\left(\frac{1}{R^d} \int_{Y_{R-R^\delta}} |\nabla(w_T^\xi(x) - w_T^{R,D,\xi}(x))|^2\, dx \right)^{1/2} \lesssim R^{d+3\delta-d\delta} \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right). \tag{5.28}
$$

Step 2 Let $\xi = e_l$ and denote the solutions of equations (5.8) and (5.6) as w_T^l and $w_T^{R,D,l}$. For $x \in Y_1$, define the functions

$$
\begin{aligned}
\tilde{w}_T^l(x) &= \frac{1}{R} w_T^l(Rx) \\
\tilde{w}_T^{R,D,l}(x) &= \frac{1}{R} w_T^{R,l}(Rx).
\end{aligned}
$$

Then these functions satisfy respectively the following equations in Y_1:

$$
\begin{aligned}
-\nabla \cdot (A\nabla \tilde{w}_T^l) + R^2 T^{-1} \tilde{w}_T^l &= \nabla A e_l, \\
-\nabla \cdot (A\nabla \tilde{w}_T^{R,D,l}) + R^2 T^{-1} \tilde{w}_T^{R,D,l} &= \nabla A e_l.
\end{aligned}
$$

Also,

$$\left.\begin{aligned} \int_{Y_1} |\nabla \tilde{w}^l_T(x)|^2\ dx &\lesssim \tfrac{1}{R^d}\int_{Y_R} |\nabla w^l_T(x)|^2\ dx \lesssim C,\\ \int_{Y_1} |\nabla \tilde{w}^{R,D,l}_T(x)|^2\ dx &\lesssim \tfrac{1}{R^d}\int_{Y_R} |\nabla w^{R,D,l}_T(x)|^2\ dx \lesssim C, \end{aligned}\right\} \tag{5.29}$$

where C is a generic constant. Now, we can obtain the required estimates.

Step 3. On using (5.28) and (5.29), we have

$$\begin{aligned}
&|e_k \cdot (\overline{A}_{T,R} - A^{R,D,*}_T)e_l|\\
&= \left| \fint_{Y_R} e_k \cdot A\nabla(w^l_T - w^{R,D,l}_T)\ dx \right|\\
&\lesssim \left| \frac{1}{R^d}\int_{Y_{R-R^\delta}} e_k \cdot A\nabla(w^l_T - w^{R,D,l}_T)\ dx \right| + \left| \frac{1}{R^d}\int_{Y_R\setminus Y_{R-R^\delta}} e_k\cdot A\nabla w^l_T\ dx\right| + \left|\frac{1}{R^d}\int_{Y_R\setminus Y_{R-R^\delta}} e_k\cdot A\nabla w^{R,D,l}_T\ dx\right|\\
&\lesssim \left| \frac{1}{R^d}\int_{Y_{R-R^\delta}} e_k \cdot A\nabla(w^l_T - w^{R,D,l}_T)\ dx \right| + \left| \int_{Y_1\setminus Y_{1-R^{\delta-1}}} e_k\cdot A\nabla \tilde{w}^l_T\ dx\right| + \left|\int_{Y_1\setminus Y_{1-R^{\delta-1}}} e_k\cdot A\nabla \tilde{w}^{R,D,l}_T\ dx\right|\\
&\lesssim \left(\frac{1}{R^d}\int_{Y_{R-R^\delta}} |\nabla(w^l_T - w^{R,D,l}_T)|^2\ dx \right)^{1/2} + \left(\int_{Y_1\setminus Y_{1-R^{\delta-1}}} |\nabla \tilde{w}^l_T|\ dx\right) + \left(\int_{Y_1\setminus Y_{1-R^{\delta-1}}} |\nabla \tilde{w}^{R,D,l}_T|\ dx\right)\\
&\lesssim \left(\frac{1}{R^d}\int_{Y_{R-R^\delta}} |\nabla(w^l_T - w^{R,D,l}_T)|^2\ dx \right)^{1/2} + \left(\int_{Y_1\setminus Y_{1-R^{\delta-1}}} |\nabla \tilde{w}^l_T|^2\ dx \int_{Y_1\setminus Y_{1-R^{\delta-1}}} 1\ dx\right)^{1/2}\\
&\qquad + \left(\int_{Y_1\setminus Y_{1-R^{\delta-1}}} |\nabla \tilde{w}^{R,D,l}_T|^2\ dx \int_{Y_1\setminus Y_{1-R^{\delta-1}}} 1\ dx\right)^{1/2}\\
&\lesssim R^{d-3\delta+d\delta}\exp\left(-c\frac{R^\delta}{\sqrt{T}}\right) + R^{(\delta-1)/2}.
\end{aligned}$$

□

5.2.9 Rate of convergence of regularized Dirichlet correctors

In this subsection, the convergence rate for the last term in (5.11) is established.

Theorem 5.11. *Let* $\rho(A, L)$ *satisfy* $\rho(A, L) \lesssim 1/L^\tau$ *for some* $\tau > 0$. *Then for any* $0 < \gamma < \frac{\tau}{\tau+1}$,

$$|A^{R,D,*} - A^{R,D,*}_T| \le C_\gamma R^{4-2\gamma}T^{-2}. \tag{5.30}$$

Proof. Observe that

$$\xi\cdot A^{R,D,*}\xi = \frac{1}{|Y_R|}\int_{Y_R} (\xi + \nabla w^{R,D,\xi})\cdot A(\xi + \nabla w^{R,D,\xi})\ dy,$$

and

$$\xi \cdot A_T^{R,D,*}\xi = \frac{1}{|Y_R|}\int_{Y_R} (\xi + \nabla w_T^{R,D,\xi}) \cdot A(\xi + \nabla w_T^{R,D,\xi})\ dy,$$

where $w^{R,D,\xi}$ solves (5.5) and $w_T^{R,D,\xi}$ solves (5.6). Hence,

$$\begin{aligned}
&\xi\cdot(A_T^{R,D,*} - A^{R,D,*})\xi \\
&\quad = \fint_{Y_R} (\xi + \nabla w_T^{R,D,\xi}) \cdot A(\xi + \nabla w_T^{R,D,\xi}) - (\xi + \nabla w^{R,D,\xi}) \cdot A(\xi + \nabla w^{R,D,\xi})\ dy \\
&\quad = \fint_{Y_R} (\xi + \nabla w_T^{R,D,\xi}) \cdot A\nabla(w_T^{R,D,\xi} - w^{R,D,\xi}) + \nabla(w_T^{R,D,\xi} - w^{R,D,\xi}) \cdot A(\xi + \nabla w^{R,D,\xi}) dy \\
&\quad = \fint_{Y_R} (\xi + \nabla w_T^{R,D,\xi}) \cdot A\nabla(w_T^{R,D,\xi} - w^{R,D,\xi}) - \nabla(w_T^{R,D,\xi} - w^{R,D,\xi}) \cdot A(\xi + \nabla w^{R,D,\xi}) dy \\
&\quad = \fint_{Y_R} \nabla(w_T^{R,D,\xi} - w^{R,D,\xi}) \cdot A\nabla(w_T^{R,D,\xi} - w^{R,D,\xi})\ dy.
\end{aligned}$$

Define $\psi_T^R = -T(w_T^{R,D,\xi} - w^{R,D,\xi})$, then the above identity becomes

$$\xi \cdot (A_T^{R,D,*} - A^{R,D,*})\xi = T^{-2}\fint_{Y_R} \nabla\psi_T^R \cdot A\nabla\psi_T^R\ dy. \tag{5.31}$$

We know that $\psi_T^R \in H_0^1(Y_R)$ solves the equation

$$T^{-1}\psi_T^R - \nabla \cdot (A\nabla\psi_T^R) = w^{R,D,\xi} \text{ in } Y_R.$$

Therefore, integrating this equation against ψ_T^R gives

$$T^{-1}\int_{Y_R} |\psi_T^R|^2\ dy + \int_{Y_R} \nabla\psi_T^R \cdot A\nabla\psi_T^R\ dy = \int_{Y_R} w^{R,D,\xi}\psi_T^R\ dy.$$

Dropping the first term on LHS yields

$$\int_{Y_R} \nabla\psi_T^R \cdot A\nabla\psi_T^R\ dy \leq \int_{Y_R} w^{R,D,\xi}\psi_T^R\ dy.$$

Hence,

$$\int_{Y_R} \nabla\psi_T^R \cdot A\nabla\psi_T^R\ dy \leq \|w^{R,D,\xi}\|_{L^2(Y_R)}\|\psi_T^R\|_{L^2(Y_R)}.$$

By coercivity of A,

$$\alpha\int_{Y_R} |\nabla\psi_T^R|^2\ dy \leq \|w^{R,D,\xi}\|_{L^2(Y_R)}\|\psi_T^R\|_{L^2(Y_R)}.$$

On applying Poincaré inequality:

$$\alpha\|\nabla\psi_T^R\|_{L^2(Y_R)}^2 \lesssim R\|w^{R,D,\xi}\|_{L^2(Y_R)}\|\nabla\psi_T^R\|_{L^2(Y_R)},$$

or

$$\|\nabla \psi_T^R\|_{L^2(Y_R)} \lesssim R\|w^{R,D,\xi}\|_{L^2(Y_R)}.$$

Substituting the above in (5.31) gives

$$\xi \cdot (A_T^{R,D,*} - A^{R,D,*})\xi \lesssim R^2 T^{-2} \fint_{Y_R} |w^{R,D,\xi}|^2 \, dy. \tag{5.32}$$

For $x \in Y_1$, define $\tilde{w}^{R,D,\xi}(x) = \frac{1}{R} w^{R,D,\xi}(Rx)$, then $\tilde{w}^{R,D,\xi}$ satisfies the equation:

$$-\nabla \cdot (A(Rx)(\xi + \nabla \tilde{w}^{R,D,\xi}(x)) = 0, \; x \in Y_1$$
$$\tilde{w}^{R,D,\xi}(x) = 0 \quad \text{on } \partial Y_1.$$

This equation is a particular case of the following homogenization problem:

$$\left.\begin{array}{ll} -\nabla \cdot A\left(\frac{x}{\epsilon}\right)(z + \nabla v^\epsilon) = h & \text{in } \tilde{\Omega}, \\ v^\epsilon = 0 & \text{on } \partial\tilde{\Omega}, \end{array}\right\} \tag{5.33}$$

where $z \in L^2(\tilde{\Omega})$, $h \in H^{-1}(\tilde{\Omega})$. By [JKO94, Theorem 5.2], the solutions v^ϵ converge weakly to v^0 in $H_0^1(\tilde{\Omega})$ which satisfies the equation

$$-\nabla \cdot A^* \left(z + \nabla v^0\right) = h, \; x \in \tilde{\Omega}.$$

Therefore, $\tilde{w}^{R,D,\xi} \rightharpoonup \tilde{w}^{D,\infty}$ in $H_0^1(Y_1)$, which satisfies the equation

$$-\nabla \cdot A^*(\xi + \nabla \tilde{w}^{D,\infty}) = 0, \; x \in Y_1.$$

The zero Dirichlet boundary condition on $\tilde{w}^{D,\infty}$ forces $\tilde{w}^{D,\infty} = 0$ a.e. Now, the analysis of proof of [She15, inequality (1.8)] shows that it remains valid for solutions of (5.33) in the following form:

$$\|v^\epsilon - v^0\|_{L^2(\tilde{\Omega})} \leq C_\gamma \epsilon^\gamma \|z + \nabla v^0\|_{H^1(\tilde{\Omega})},$$

for any $0 < \gamma < \frac{\tau}{\tau+1}$. Therefore,

$$\|\tilde{w}^{R,D,\xi}\|_{L^2(Y_1)} = \|\tilde{w}^{R,D,\xi} - \tilde{w}^{D,\infty}\|_{L^2(Y_1)} \leq C_\gamma R^{-\gamma},$$

for any $0 < \gamma < \frac{\tau}{\tau+1}$. Finally, it follows that

$$\begin{aligned} \xi \cdot (A_T^{R,D,*} - A^{R,D,*})\xi &\lesssim R^4 T^{-2} \fint_{Y_R} |w^{R,D,\xi}(x)|^2 \, dx \\ &\lesssim R^4 T^{-2} \int_{Y_1} |\tilde{w}^{R,D,\xi}(x)|^2 \, dx \\ &\lesssim R^{4-2\gamma} T^{-2}. \end{aligned} \tag{5.34}$$

□

5.2.10 Proof of Theorem 5.3

Proof. Using theorems 5.5, 5.8, 5.10, 5.11 and the inequality (5.11), we obtain

$$|A^* - A^{R,D,*}| \lesssim \frac{1}{T^{\frac{\tau}{2(\tau+1)}-\omega}} + \frac{1}{L^\tau} + \left(\frac{L}{R}\right)^{1/2} + R^d \exp\left(-c\frac{R^\delta}{\sqrt{T}}\right) + \frac{1}{R^{(1-\delta)/2}} + R^{4-2\gamma}T^{-2}. \tag{5.35}$$

Let $\gamma', \beta_1 \in (0,1)$. By choosing $\gamma' = \gamma/2$, $T = R^{2-\gamma'}$, $L = R^{\beta_1}$, $\beta_2 = 2\gamma'$ and $\delta = 1 - \beta_2/8$, we can obtain the estimate $|A^* - A^{R,D,*}| \lesssim \frac{1}{R^\beta}$, for some $\beta > 0$. □

Remark 5.12.

1. The main difficulty for obtaining rate of convergence estimate for the approximate homogenized tensor $A^{R,*}$ corresponding to the periodization A^R is the absence of rate of convergence for almost periodic homogenization of periodic boundary value problems. Recall that Shen [She15] has obtained rate of convergence estimates for almost periodic homogenization of Dirichlet boundary value problems. These are used in the proof of Theorem 5.11.

2. Another way to obtain the convergence estimate for the approximate homogenized tensor $A^{R,*}$ corresponding to the periodization A^R would be to show that $\rho(A_R, L) \sim \rho(A, L)$. Indeed, in approximations of homogenized tensors for stochastic media, it is typically assumed that the probability distribution of the coefficients on every cube of side length $2\pi L$ coincides with the probability distribution of the original coefficient field [Fis19].

3. Rate of convergence for periodic homogenization of periodic boundary value problems can be obtained as suggested in [JKO94, p. 30]. However, the mismatch of periodic boundary conditions and almost periodic media appears to be a difficult problem.

4. The above considerations also suggest another question, whether the Dirichlet and Periodic correctors grow close to each other in the limit of $R \to \infty$. Mathematically, we may ask an estimate for $\left(⨍_{Y_R} |\nabla w^{R,D,\xi}(y) - \nabla w^{R,\xi}(y)|^2\, dy\right)^{1/2}$.

5.3 Numerical study

In this section, we report on the numerical experiments that we carried out for certain benchmark periodic and quasiperiodic functions introduced in [Glo11, GH16]. It is known that approximations of homogenized tensor for periodic media using Dirichlet and Periodic correctors have a rate of convergence of R^{-1} [AAP19, Cor. 1]. Our aim is to verify such results. We also numerically study the difference of Dirichlet and Periodic correctors as we feel that this difference should also show decay. These computations are done using the finite element method on FEniCS software [ABH+15].

5.3.1 Numerical study for Dirichlet Approximations

In this subsection, we investigate the behavior of the error in the Dirichlet approximations $|A^{R,D,*} - A^*|$ with respect to side length R.

The first two examples are that of periodic matrices

$$A_1(x) = \left(\frac{2 + 1.8\sin(2\pi x)}{2 + 1.8\cos(2\pi y)} + \frac{2 + \sin(2\pi y)}{2 + 1.8\cos(2\pi x)}\right) \mathrm{Id}, \text{ and}$$

$$A_2(x) = (1 + 30(2 + \sin(2\pi x)\sin(2\pi y)))\,\mathrm{Id}\,.$$

The homogenized tensor A^* is computed numerically by solving the periodic cell problem on the unit cube $[0, 1)^d$ and is found to be approximately $2.757\,\mathrm{Id}$ and $59.1\,\mathrm{Id}$ for A_1 and A_2 respectively. The approximate homogenized tensor $A^{R,D,*}$ is computed by solving the Dirichlet cell problem (5.5) for different values of R going up to 40. The computations are carried out with P2-Finite Elements discretization and 20 points per dimension in every unit cell. See Figure 5.1 for the log-log plot of the error $|A^{R,D,*} - A^*|$ with respect to R.

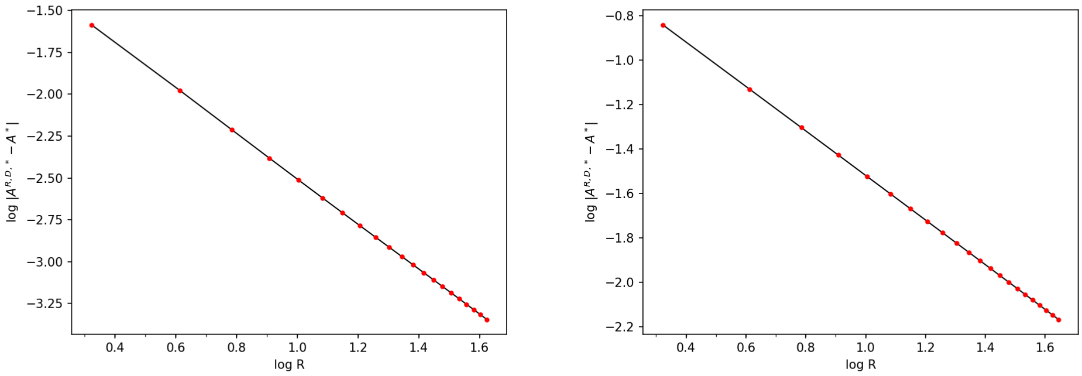
The third example is that of the following matrix with quasiperiodic entries:

$$A_3(x) = \left(4 + \cos(2\pi(x + y)) + \cos(2\pi\sqrt{2}(x + y))\right) \mathrm{Id}$$

The homogenized coefficient for quasiperiodic media A^* (4.61) is defined as a mean value in the full space $\mathbb{R}^d$ and therefore it is impossible to compute. Hence, for the computation of the error, A^* is taken to be the value of $A_T^{R,D,*}$ (5.10) for $R = T = 60$, since $A_T^{R,D,*}$ is known to converge faster to A^* as $R, T \to \infty$ [Glo11, GH16]. See Figure 5.2 for the log-log plot of the error $|A^{R,D,*} - A^*|$ with respect to R.

5.3.2 Numerical study for approximations of A^* using periodic correctors

In this subsection, we investigate the behavior of the error in approximations to homogenized tensor $|A^{R,*} - A^*|$ using periodic correctors with respect to R.



(a) Periodic function A_1

(b) Periodic function A_2

Figure 5.1: The error $|A^{R,D,*} - A^*|$ for Dirichlet approximations in log-log scale for the functions A_1 and A_2 with respect to R.

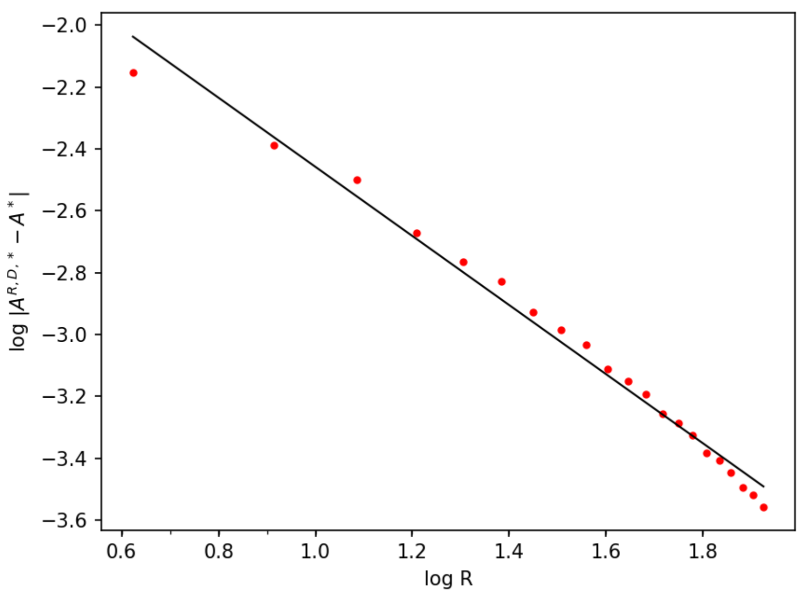


Figure 5.2: The error $|A^{R,D,*} - A^*|$ for Dirichlet approximations in log-log scale for the function A_3 with respect to R.

The approximate homogenized tensor $A^{R,*}$ corresponding to periodization A^R has already been defined in (4.19).

Although we have been unable to establish a rate of convergence in this case, the log-log plots of errors in periodic and quasiperiodic cases seem to suggest an asymptotically polynomial rate of convergence. Computations are performed with P1 finite elements with a varying choice of number of meshpoints n per dimension, as denoted in Figures 5.3 and 5.4.

5.3.3 Comparison of Dirichlet and Periodic Correctors

An interesting question that arises in Section 5.2 is whether the Dirichlet and periodic correctors, respectively $w^{R,D,\xi}$ and $w^{R,\xi}$, grow close to each other as the side length R of sample cube increases. It is evident that the two approximations satisfy the same differential equation in the interior of the cube and only differ in the boundary conditions. An attempt to prove an estimate for $E(R) = \left(⨍_{Y_R} |\nabla w^{R,D,e_1}(y) - \nabla w^{R,e_1}(y)|^2 \, dy \right)^{1/2}$ using Green's function estimate seems to fail. However, the regularized versions of the problems can be shown to have an asymptotic rate of convergence of any order due to exponential decay of Green's function of the operator $-\nabla \cdot A\nabla + T^{-1}$ with Dirichlet boundary conditions.

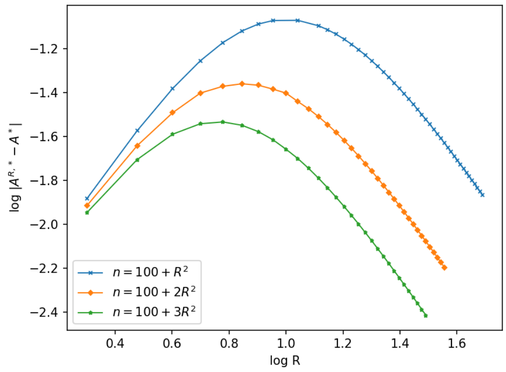
In Figure 5.5, we plot the error $E(R)$ with respect to R on a log-log scale for functions A_1 and A_2. In Figure 5.6, we plot the error $E(R)$ with respect to R on a log-log scale for A_3. The numerical study is carried out with P1 finite elements. The number of meshpoints per dimension is taken to be $n = 100 + R^2$.

In Figure 5.7, we plot the error $|A^{R,D,*} - A^{R,*}|$ with respect to R on a log-log scale for functions A_1 and A_2. In Figure 5.8, we plot the error $|A^{R,D,*} - A^{R,*}|$ with respect to R on a log-log scale for A_3.

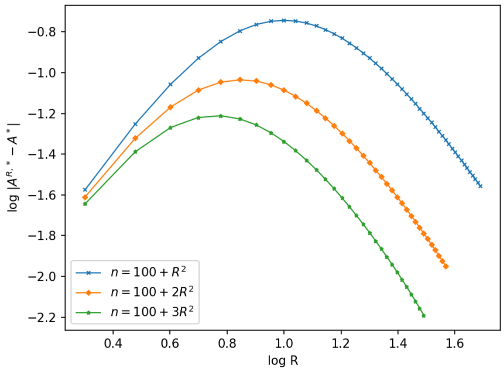
5.4 Comments

The methods that are used in this chapter are physical space methods that are mostly due to Shen [She15]. We would like to explore some spectral theory methods for this problem.

We also wonder whether the estimate which is obtained is sharp. A theorem of [BBMM05] asserts that the rate of convergence in almost periodic homogenization can be as slow as desired. Since our proof relies on rate of convergence in almost periodic homogenization, it is doubtful that it can be improved. On the other hand, it is already well known that for media satisfying Kozlov's small divisors condition,



(a) Periodic function A_1



(b) Periodic Function A_2

Figure 5.3: The error $|A^{R,*} - A^*|$ for approximations to homogenized tensor using periodic correctors in log-log scale for the functions A_1 and A_2 with respect to R

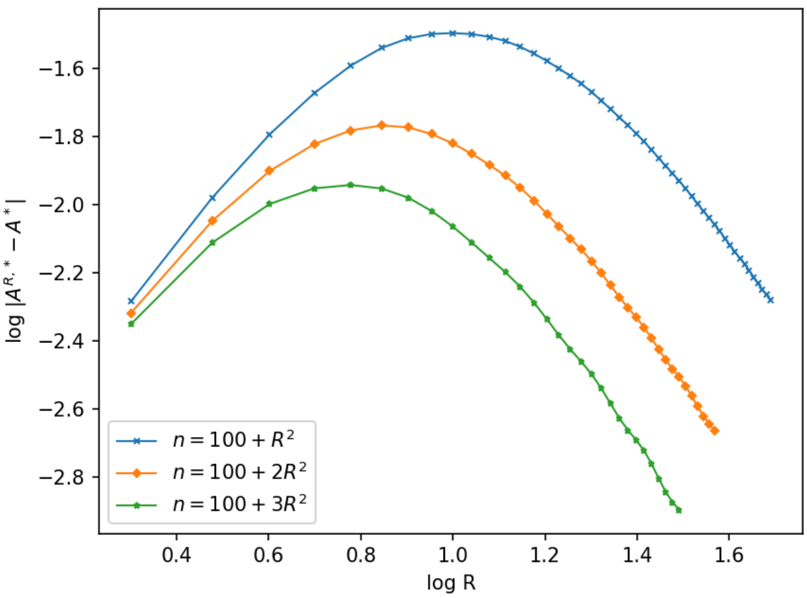
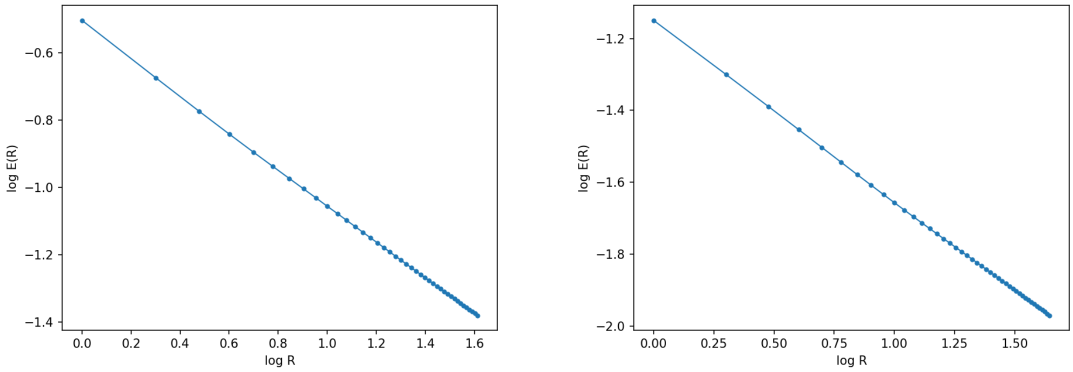


Figure 5.4: The error $|A^{R,*} - A^*|$ for approximations to homogenized tensor using periodic correctors in log-log scale for the function A_3 with respect to R



(a) Periodic function A_1 (b) Periodic function A_2

Figure 5.5: The averaged L^2 norm of the difference of the gradients $E(R) = \left(\fint_{Y_R} |\nabla w^{R,D,e_1}(y) - \nabla w^{R,e_1}(y)|^2\, dy\right)^{1/2}$ in log-log scale for the correctors corresponding to the periodic matrices A_1 and A_2 plotted as a function of R.

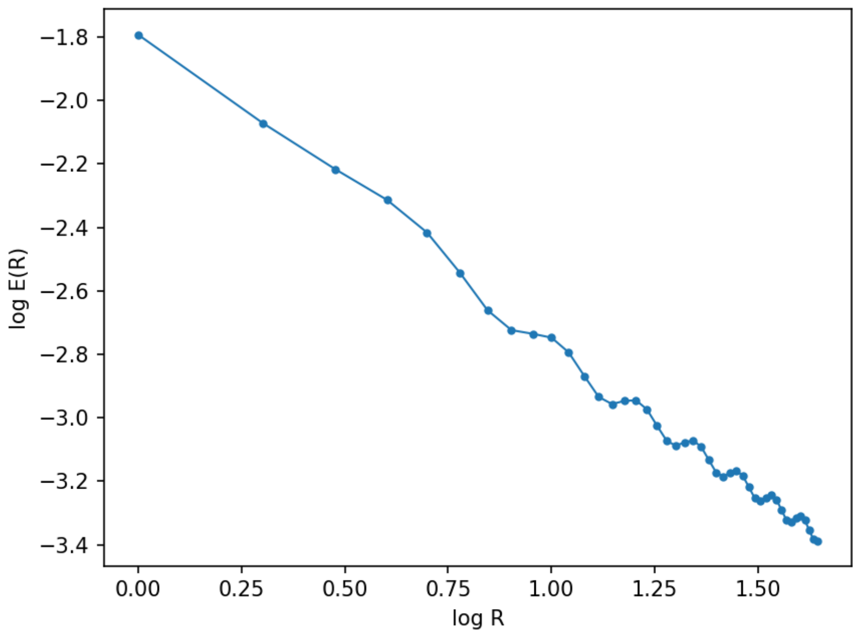
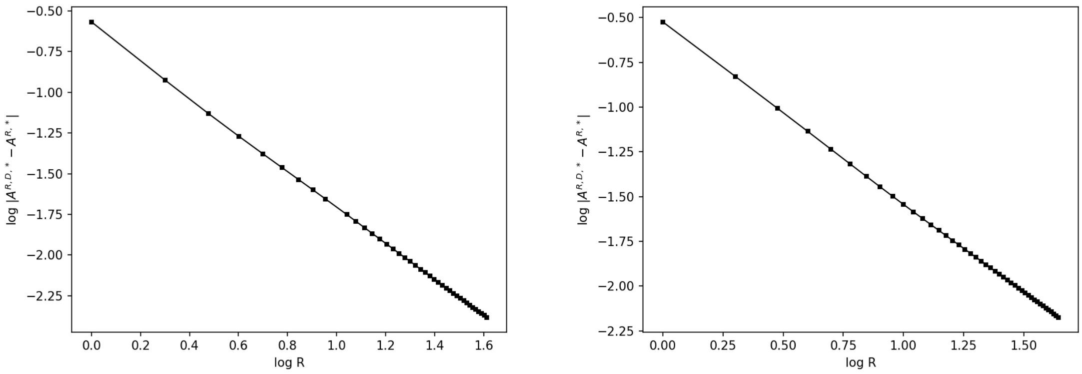


Figure 5.6: The averaged L^2 norm of the difference of the gradients $E(R) = \left(\fint_{Y_R} |\nabla w^{R,D,e_1}(y) - \nabla w^{R,e_1}(y)|^2\, dy\right)^{1/2}$ in log-log scale for the correctors corresponding to the quasiperiodic matrix A_3 plotted as a function of R.



(a) Periodic function A_1 (b) Periodic function A_2

Figure 5.7: The absolute error $|A^{R,D,*} - A^{R,*}|$ in log-log scale for the periodic matrices A_1 and A_2 plotted as a function of R.

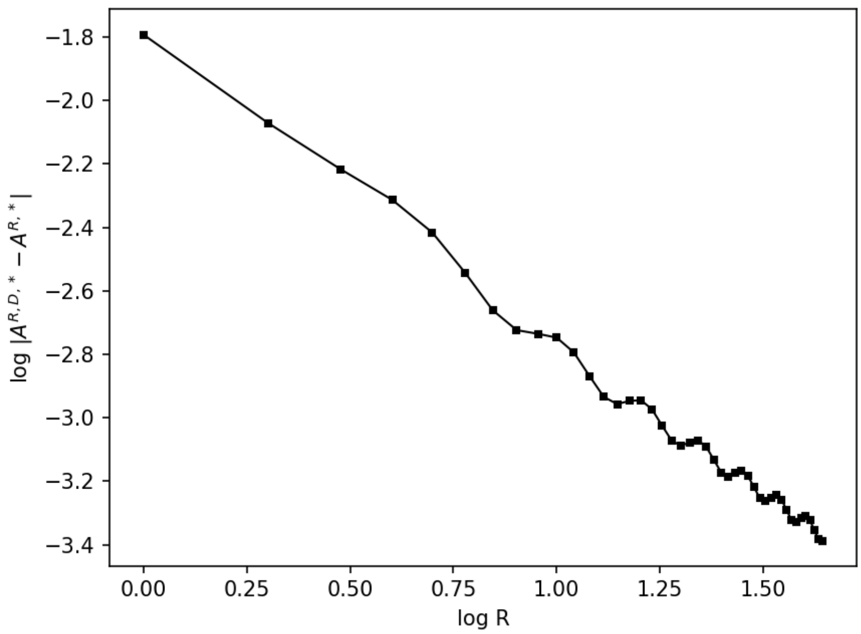


Figure 5.8: The absolute error $|A^{R,D,*} - A^{R,*}|$ in log-log scale for the quasiperiodic matrix A_3 plotted as a function of R.

the rate of convergence is as good as the one for periodic homogenization [GH16]. Hence, the rate of convergence obtained in this thesis is not optimal for quasiperiodic media satisfying the Kozlov condition.

Chapter 6

Bloch wave homogenization of quasiperiodic media

Quasiperiodic media is a class of almost periodic media which is generated from periodic media through a "cut and project" procedure. Bloch waves are typically defined through a direct integral decomposition of periodic operators. A suitable direct integral decomposition is not available for almost periodic operators. To remedy this, we lift an almost periodic operator to a degenerate periodic operator in higher dimensions. Approximate Bloch waves are obtained for a regularized version of the degenerate operator. Homogenized coefficients for quasiperiodic media are obtained from the first Bloch eigenvalue of the regularized operator in the limit of regularization parameter going to zero. A notion of quasiperiodic Bloch transform is defined and employed to obtain homogenization limit for an equation with highly oscillating quasiperiodic coefficients.

6.1 Introduction

In this chapter, we will perform Bloch wave homogenization of the following equation with highly oscillatory quasiperiodic coefficients:

$$-\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}(x) = f \text{ in } \Omega$$
$$u^{\varepsilon} = 0 \text{ on } \partial\Omega, \tag{6.1}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded domain. Bloch wave homogenization is a framework developed by Conca and Vanninathan [CV97] for obtaining qualitative as well as quantitative results in periodic homogenization. Further, Bloch decomposition has been employed by Birman and Suslina [BS04] to obtain order-sharp estimates for systems in the theory of homogenization with minimal regularity requirements. Let M be an integer such that $M > d$ and let $Q = [0, 2\pi)^M$ denote a parametrization of the M-dimensional torus $\mathbb{T}^M$. We make the following assumptions on the coefficient matrix $A = (a_{kl})_{k,l=1}^d$:

(H1) The entries $(a_{kl})_{k,l=1}^d$ are smooth, bounded real-valued functions defined on $\mathbb{R}^d$.

(H2) The coefficient matrix A is quasiperiodic, i.e., there exists a $d \times d$ matrix B with smooth Q-periodic entries and a constant $M \times d$ matrix Λ such that $A = B \circ \Lambda$, i.e.,

$$\forall\, x \in \mathbb{R}^d \text{ and } \forall\, k, l \text{ s.t. } 1 \leq k, l \leq d \quad a_{kl}(x) = b_{kl}(\Lambda x),$$

where the matrix Λ satisfies

$$\Lambda^T p \neq 0 \text{ for non-zero } p \in \mathbb{Z}^M. \tag{6.2}$$

(H3) The matrix A is symmetric.

(H4) The matrix A is coercive, i.e., there is a positive real number α such that for all $v \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$, we have

$$\langle A(x)v, v\rangle \geq \alpha |v|^2.$$

Remark 6.1.

1. The assumption of smoothness on the entries of A is not essential. The approach of this chapter demands taking trace of solutions on lower dimensional manifolds. We only require as much smoothness as would guarantee twice continuous differentiability of the solutions.

2. The assumption (6.2) implies that the continuous and periodic matrix B is uniquely determined from its values on $\Lambda\mathbb{R}^d$. Hence, coercivity of B on $\mathbb{R}^M$ follows from that of A. See [BCPD92, Section 3] for proof.

The class of quasiperiodic functions is a subclass of almost periodic functions. For $K = \mathbb{R}$ or $\mathbb{C}$, let $\mathrm{Trig}(\mathbb{R}^d; K)$ denote the set of all K-valued trigonometric polynomials. Recall that the completion of $\mathrm{Trig}(\mathbb{R}^d; K)$ in norm of uniform convergence results in a Banach space called the space of all Bohr almost periodic functions denoted as $AP(\mathbb{R}^d)$. Further, in $L^p_{loc}(\mathbb{R}^d)$, one can define a seminorm

$$\|f\|_{B^p} := \left(\limsup_{R\to\infty} \frac{1}{R^d}\int_{[-\frac{R}{2},\frac{R}{2})^d} |f(y)|^p\,dy\right)^{1/p}.$$

For $1 \le p < \infty$, the completion of $\mathrm{Trig}(\mathbb{R}^d;K)$ in this seminorm results in the Besicovitch space of almost periodic functions $B^p(\mathbb{R}^d)$. Given a Besicovitch almost periodic function g, one can define the notion of mean value

$$\mathcal{M}(g) := \lim_{R\to\infty} \frac{1}{R^d}\int_{[-\frac{R}{2},\frac{R}{2})^d} g(y)\,dy.$$

For each $g \in B^p(\mathbb{R}^d)$, we can associate a formal Fourier series $g \sim \sum_{\xi\in\mathbb{R}^d} \widehat{g}(\xi)e^{ix\cdot\xi}$, whose exponents are those vectors $\xi \in \mathbb{R}^d$ such that $\mathcal{M}(g\cdot\exp(ix\cdot\xi)) \neq 0$. These exponents or frequencies are denoted by $\exp(g)$ and the $\mathbb{Z}$-module generated by $\exp(g)$ is called as the frequency module of g and denoted by $\mathrm{Mod}(g)$. A *quasiperiodic* function may also be defined as an almost periodic function whose frequency module is finitely generated (See 2 in Remark 6.2). Trigonometric polynomials are the most common example of quasiperiodic functions. One may conclude from this definition that any quasiperiodic function may be lifted through a *winding matrix* Λ to a periodic function on a higher dimensional torus. The space of all periodic L^2 functions in the higher dimension will be denoted interchangeably by $L^2_\sharp(Q)$ or $L^2(\mathbb{T}^M)$. The space $L^2_\sharp(Q)$ is also defined as the closure of $C^\infty_\sharp(Q)$ functions in $L^2(Q)$ norm. Similarly, for $s \in \mathbb{R}$, we may define $H^s_\sharp(Q)$ or $H^s(\mathbb{T}^M)$ as the space of all periodic distributions for which the norm $\|u\|_{H^s} = \left(\sum_{n\in\mathbb{Z}^M}(1+|n|^2)^s|\widehat{u}(n)|^2\right)^{1/2}$ is finite.

Remark 6.2.

1. The assumption (6.2) makes sure that the mean value of the quasiperiodic matrix A can be written as the mean value of the periodic matrix B on Q. A proof of this fact may be found in [Shu78]. The equality of the two mean values is used in Section 6.6 for the characterization of homogenized tensor of quasiperiodic media.

2. We have given two seemingly disparate definitions of quasiperiodic functions, one as restriction of periodic functions to lower dimensional planes and second through the frequency module. Indeed, the two definitions are equivalent and the proof may be found in [BP01] for different classes of almost periodic

functions. Let $\Gamma \subseteq \mathbb{R}^d$ be a finitely generated $\mathbb{Z}$-module. Denote by $B^2_\Gamma(\mathbb{R}^d)$ (respectively $AP_\Gamma(\mathbb{R}^d)$) the subspace of $B^2(\mathbb{R}^d)$ (respectively $AP(\mathbb{R}^d)$) containing functions whose frequencies belong to Γ. Then, $B^2_\Gamma(\mathbb{R}^d)$ (respectively $AP_\Gamma(\mathbb{R}^d)$) is isometrically isomorphic to $L^2(\mathbb{T}^N)$ (respectively $C(\mathbb{T}^N)$) for some $N > d$.

3. A simple example of a quasiperiodic function is $g(x) = \sin(x) + \sin(\sqrt{2}x)$ which admits a periodic embedding of the form $\tilde{g}(x, y) = \sin(x) + \sin(y)$ with $\Lambda = (1 \quad \sqrt{2})^T$. One may wonder how to obtain a common matrix Λ for a collection of functions such as in the case of the entries of a quasiperiodic matrix. This is not too difficult either, as illustrated in the following example. Consider the quasiperiodic matrix

$$A = \begin{pmatrix} \sin(x) + \sin(\sqrt{2}x) & \cos(\sqrt{2}x) \\ \cos(\sqrt{2}x) & \cos(\sqrt{3}x) \end{pmatrix}, x \in \mathbb{R}.$$

The matrix A admits the following periodic embedding.

$$B = \begin{pmatrix} \sin(x) + \sin(y) & \cos(y) \\ \cos(y) & \cos(z) \end{pmatrix}, (x, y, z) \in \mathbb{R}^3,$$

with $\Lambda = (1 \quad \sqrt{2} \quad \sqrt{3})^T$.

4. The assumption (6.2) is a qualitative version of Kozlov's small divisors condition which we recall below.

Definition 6.3. A quasiperiodic function $f : \mathbb{R}^d \to \mathbb{R}$ is said to satisfy the **Kozlov condition** if

(1) there exist a function $F : \mathbb{R}^M \to \mathbb{R}$ and $M(= m_1 + m_2 + \ldots + m_d)$ numbers $\beta_1^1, \beta_1^2, \ldots, \beta_1^{m_1}, \beta_2^1, \ldots, \beta_2^{m_2}, \ldots, \beta_d^1, \ldots, \beta_d^{m_d} \in \mathbb{R}$ such that

$$f(x) = F\left(\beta_1^1 x_1, \beta_1^2 x_1, \ldots, \beta_1^{m_1} x_1, \beta_2^1 x_2, \ldots, \beta_2^{m_2} x_2, \ldots, \beta_d^1 x_d, \ldots, \beta_d^{m_d} x_d\right).$$

(2) For each $1 \leq i \leq d$, $\beta_i := (\beta_i^1, \beta_i^2, \ldots, \beta_i^{m_i})$ is linearly independent over $\mathbb{Z}$.

(3) There exist $C > 0$ and $\tau > 0$ such that for each $i = 1, 2, \ldots, d$ such that $m_i \geq 2$,

$$|n \cdot \beta_i| \geq \frac{C}{|n|^\tau}, \text{ for all } n \in \mathbb{Z}^{m_i} \setminus \{0\}. \tag{6.3}$$

A standard method to solve equations with quasiperiodic coefficients is to propose and solve an equation in higher dimensions whose solutions when suitably restricted to $\mathbb{R}^d$ solve the original equation [Koz78, GH16, BLBL15, WGC18]. Such a procedure necessitates an assumption on coefficients to be at least continuous since restriction of functions to lower dimensional surfaces requires some smoothness. A second difficulty results from the fact that the equation posed in the higher dimension is typically degenerate or non-elliptic. In order to define a suitable notion of Bloch waves, we regularize the degenerate equation in higher dimension. The homogenized tensor for quasiperiodic media is found to be equal to the limit of the Hessian of first Bloch eigenvalue of the regularized degenerate operator as the regularization parameter tends to zero. Further, we define a notion of quasiperiodic Bloch transform to aid us in the passage to the homogenization limit.

We note here that the study of almost periodic homogenization was initiated by Kozlov [Koz78] who also obtained a rate of convergence for quasiperiodic media satisfying a small divisors condition called the Kozlov condition. A widely known example of quasiperiodic media is quasicrystals [SBGC84]. Quasicrystals are ordered structures without periodicity. They may be thought of as periodic crystals in higher dimensions that are projected to lower dimensions through a "cut and project" procedure. Quasicrystals have unique thermal and electrical conductivity properties with many potential industrial and household applications, such as adhesion and friction resistant agents, composite materials [Dub12]. The mathematical structure of quasicrystals had already been anticipated in the works of Bohr [Boh47], Besicovitch [Bes55], and Meyer [Mey95].

The plan of the chapter is as follows: In Section 6.2, we introduce the degenerate periodic equation in $\mathbb{R}^M$ and its regularized version for which we obtain approximate Bloch waves. In Section 6.3, we prove the existence of the regularized Bloch waves. In Section 6.4, we apply Kato-Rellich theorem to obtain analytic branch of the regularized Bloch waves and Bloch eigenvalue. In Section 6.5, we recall the cell problem for almost periodic media and the cell problem for the degenerate periodic operator in higher dimensions. In Section 6.6, we obtain the homogenized tensor for the quasiperiodic media as a limit of the first regularized Bloch eigenvalue. In Section 6.7, we introduce a notion of quasiperiodic Bloch transform. Finally, in Section 6.8, we obtain the homogenization theorem for quasiperiodic media by using the quasiperiodic Bloch transform.

The contents of this chapter form a section of the preprint [3].

6.2 Degenerate operator in $\mathbb{R}^M$

The Bloch wave method in homogenization is a spectral method. Bloch waves are solutions to the Bloch spectral problem which is a parametrized eigenvalue problem. While the details of Bloch wave method can be found in [CV97], the main feature of this method is the existence of a "ground state" for the periodic operator, which is facilitated by the direct integral decomposition of the periodic operator. In the case of a quasiperiodic operator, one may not have a ground state but we show the existence of an approximate ground state. To begin with, we shall pose a Bloch spectral problem for the quasiperiodic operator. Let $Y' := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$, then we seek quasiperiodic solutions to the following Bloch spectral problem for the quasiperiodic operator $\mathcal{A} = -\nabla \cdot (A\nabla)$

$$-(\nabla + i\eta) \cdot A(\nabla + i\eta)\phi = \lambda\phi \text{ in } \mathbb{R}^d. \tag{6.4}$$

The problem above is typically solved for periodic A, in which case, the solutions are called Bloch waves. However, the matrix A is quasiperiodic, and it is not clear whether quasiperiodic solutions to (6.4) exist. Therefore, we propose to lift the operator $\mathcal{A}$ to a periodic operator in $\mathbb{R}^M$, for which a functional analytic formalism is available. The mapping $x \mapsto \Lambda x \in \mathbb{R}^M$ lifts the operator $\mathcal{A}$ to the periodic but degenerate operator in $\mathbb{R}^M$ given by

$$\mathcal{C} := -\Lambda^T\nabla_y \cdot B\Lambda^T\nabla_y. \tag{6.5}$$

Let us denote $\Lambda^T\nabla_y$ by D, then operator $\mathcal{C}$ is written as $-D \cdot BD$. The operator $\mathcal{C}$ may also be written as $-\nabla_y \cdot C\nabla_y$ where the matrix $C = \Lambda B\Lambda^T$. Note that C is non-coercive.

The Bloch eigenvalue problem given by (6.4) is lifted to the following problem: For $\eta \in Y'$, find $\phi(\eta) \in H^1_\sharp(Q)$ such that

$$\mathcal{C}(\eta)\phi(\eta) := -(D + i\eta) \cdot B(D + i\eta)\phi(\eta) = \lambda(\eta)\phi(\eta). \tag{6.6}$$

We note here that due to the degeneracy of operator $\mathcal{C}(\eta)$, we cannot seek Lax-Milgram solutions to this equation in $H^1_\sharp(Q)$. To remedy this situation, inspired by [BLBL15], we regularize (6.6) as follows. For $\eta \in Y'$ and $0 < \delta < 1$, find $\phi^\delta(\eta) \in H^1_\sharp(Q)$ such that

$$\mathcal{C}^\delta(\eta)\phi^\delta(\eta) := -(D + i\eta) \cdot B(D + i\eta)\phi^\delta(\eta) + \delta\Delta\phi^\delta(\eta) = \lambda^\delta(\eta)\phi^\delta(\eta). \tag{6.7}$$

The solutions ϕ^δ to (6.7) shall be called regularized Bloch waves and λ^δ will be called regularized Bloch eigenvalues.

Remark 6.4.

1. In homogenization, one often assumes the basic periodicity cell to be rectangular for convenience. However, more general periodicity cells in the shape of a parallelopiped may be considered through a change of coordinates. Under the change of coordinates, the rectangular cell becomes a parallelopiped and an operator of the form $-\nabla \cdot A\nabla$ becomes $-\nabla \cdot (PAP^{-1})\nabla$. In a similar fashion, the transformation Λ converts the operator $-\nabla_x \cdot A\nabla_x$ into the operator $-\nabla_y \cdot (\Lambda B \Lambda^T)\nabla_y$. Unlike PAP^{-1}, the matrix $\Lambda B \Lambda^T$ is non-invertible since Λ is a transformation between spaces of different dimensions.

2. It is instructive to compare quasiperiodic structures with laminates. Quasiperiodic media admit embeddings in higher-dimensions which are periodic and non-homogeneous in all directions. On the other hand, laminated materials are periodic structures which are homogeneous in some directions. Further, the operator with quasiperiodic coefficients has a degenerate embedding in higher dimensions, viz., it is non-elliptic in certain directions. On the other hand, the operator modelling laminates are elliptic in all directions.

3. The regularization may be thought of as the addition of complementary directions to the quasicrystal which is produced by "cutting" a periodic crystal in certain "irrational" directions and then projecting to lower dimensions.

4. In contrast with (6.7), it is standard to take the quasimomentum parameter η in $\mathbb{R}^M$ and to seek the regularized Bloch eigenvalues corresponding to the periodic operator given by $-\nabla_y \cdot (\Lambda B \Lambda^T + \delta I)\nabla_y$ where I is the $M \times M$ identity matrix. However, we have chosen the quasimomentum parameter η in $\mathbb{R}^d$ and we have not introduced a shift in the regularized term $\delta\Delta$. This simplifies the presentation considerably.

5. A notion of approximate Bloch waves for aperiodic media has been introduced in a dynamic context in the works of Gloria and his collaborators [BG19, DGS18]. They employ a concept of Taylor-Bloch waves for long-time homogenization of aperiodic wave equation. While our method relies on spectral theory, they make use of regularized cell problems to define Taylor-Bloch waves.

6.3 Regularized Bloch waves

In what follows, we shall prove that

1. There exists C_* such that for all $\eta \in Y'$, the bilinear form generated by the operator $\mathcal{C}^\delta(\eta)+C_*I$ is elliptic on $H^1_\sharp(Q)$ where I denotes the identity operator on $L^2_\sharp(Q)$. This will allow us to prove invertibility of $\mathcal{C}^\delta(\eta)+C_*I$.

2. By Rellich compactness theorem, we will prove compactness of the inverse of $\mathcal{C}^\delta(\eta)+C_*I$ in $L^2_\sharp(Q)$. This will prove the existence of regularized Bloch eigenvalues and Bloch eigenfunctions.

3. An application of the perturbation theory will provide us with smoothness of regularized Bloch eigenvalues and Bloch waves with respect to η near $\eta = 0$.

For the bilinear form $a^\delta[\eta](\cdot,\cdot)$ defined on $H^1_\sharp(Q)\times H^1_\sharp(Q)$ by

$$a^\delta[\eta](u,v) := \int_Q B(D+i\eta)u\cdot\overline{(D+i\eta)v}\,dy + \delta\int_Q \nabla_y u\cdot\overline{\nabla_y v}\,dy, \tag{6.8}$$

we have the following Gårding-type inequality whose proof is simple and is omitted.

Lemma 6.5. *There exist positive real numbers C_* and C^* not depending on δ and η such that for all $u \in H^1_\sharp(Q)$ and all $\eta \in Y'$, we have*

$$a[\eta](u,u) + C_*||u||^2_{L^2_\sharp(Q)} \geq \delta||\nabla_y u||^2_{L^2_\sharp(Q)} + C^*||Du||_{L^2_\sharp(Q)}. \tag{6.9}$$

The above lemma shows that for every $\eta \in Y'$ the operator $\mathcal{C}^\delta(\eta)+C_*I$ is elliptic on $H^1_\sharp(Q)$. Hence, for $f \in L^2_\sharp(Q)$, this shows that $\mathcal{C}^\delta(\eta)u + C_*u = f$ is solvable and the solution is in $H^1_\sharp(Q)$. As a result, the solution operator $S(\eta)$ is continuous from $L^2_\sharp(Q)$ to $H^1_\sharp(Q)$. Since the space $H^1_\sharp(Q)$ is compactly embedded in $L^2_\sharp(Q)$, $S(\eta)$ is a self-adjoint compact operator on $L^2_\sharp(Q)$. Therefore, by an application of the spectral theorem for self-adjoint compact operators, for every $\eta \in Y'$ we obtain an increasing sequence of eigenvalues of $\mathcal{C}^\delta(\eta)+C_*I$ and the corresponding eigenfunctions form an orthonormal basis of $L^2_\sharp(Q)$. However, note that both the operators $\mathcal{C}^\delta(\eta)$ and $\mathcal{C}^\delta(\eta)+C_*I$ have the same eigenfunctions but each eigenvalue of the two operators differ by C_*. We shall denote the eigenvalues and eigenfunctions of the operator $\mathcal{C}^\delta(\eta)$ by $\eta \to (\lambda^\delta_m(\eta), \phi^\delta_m(\cdot,\eta))$. Note that due to the regularity of the coefficients, the eigenfunctions are C^∞ functions of $y \in Q$. All of these developments are recorded in the theorem below.

Theorem 6.6. *The regularized Bloch eigenvalue problem* (6.7) *admits a countable sequence of eigenvalues and corresponding eigenfunctions in the space $H^1_\sharp(Q)$. Further, the eigenfunctions $\phi_m(y,\eta)$ are C^∞ functions of $y \in Q$.*

Proof. We have already proved the existence of the eigenvalues and eigenfunctions for the problem (6.7). Regularity of the eigenfunctions follows from the standard elliptic regularity theory [LU68]. □

Remark 6.7. In (H1), we assume the coefficient matrix A to be smooth. However, we do not require this much regularity. We only require as much smoothness on the coefficient matrix that would ensure that the Bloch eigenfunctions are twice continuously differentiable.

6.4 Regularity of the ground state

In the sequel, differentiability properties of regularized Bloch eigenvalues and regularized Bloch eigenfunctions with respect to the dual parameter $\eta \in Y'$ are required. For this purpose, we have Kato-Rellich theorem [Kat95] which guarantees analyticity of parametrized eigenvalues and eigenfunctions corresponding to analytic family of operators near a point at which the eigenvalue is simple. Indeed, we will prove the following theorem.

Theorem 6.8. *For every $\delta > 0$, there exists $\theta_\delta > 0$ and a ball $U^\delta := B_{\theta_\delta}(0) := \{\eta \in Y' : |\eta| < \theta_\delta\}$ such that*

1. *The first regularized Bloch eigenvalue $\eta \to \lambda_1^\delta(\eta)$ is analytic for $\eta \in U^\delta$.*
2. *There is a choice of corresponding eigenfunctions $\phi_1^\delta(\cdot,\eta)$ such that $\eta \in U^\delta \to \phi_1^\delta(\cdot,\eta) \in H^1_\sharp(Q)$ is analytic.*

The proof will require the Kato-Rellich theorem which we will state below for completeness. The theorem as stated in [RS78] is for a single parameter, however the theorem is also true for multiple parameters with the assumption of simplicity (See Supplement of [Bau85]).

Theorem 6.9. *(Kato-Rellich) Let $D(\tilde{\eta})$ be a self-adjoint holomorphic family of type (B) defined for $\tilde{\eta}$ in an open set in $\mathbb{C}^M$. Further let $\lambda_0 = 0$ be an isolated eigenvalue of $D(0)$ that is algebraically simple. Then there exists a neighborhood $R_0 \subseteq \mathbb{C}^M$ containing 0 such that for $\tilde{\eta} \in R_0$, the following holds:*

1. *There is exactly one point $\lambda(\tilde{\eta})$ of $\sigma(D(\tilde{\eta}))$ near $\lambda_0 = 0$. Also, $\lambda(\tilde{\eta})$ is isolated and algebraically simple. Moreover, $\lambda(\tilde{\eta})$ is an analytic function of $\tilde{\eta}$.*

2. *There is an associated eigenfunction* $\phi(\tilde{\eta})$ *depending analytically on* $\tilde{\eta}$ *with values in* $H^1_\sharp(Q)$.

In order to prove Theorem 6.8, we need to complexify the shifted operator $\mathcal{C}^\delta(\eta)$ before verifying the hypothesis of Kato-Rellich Theorem.

Proof. (Proof of Theorem 6.8)

(I) COMPLEXIFICATION OF $\mathcal{C}^\delta(\eta)$ The form $a[\eta](\cdot,\cdot)$ is associated with the operator $\mathcal{C}^\delta(\eta)$. We define its complexification as

$$t(\tilde{\eta}) = \int_Q B(D + i\sigma + \tau)u \cdot (D - i\sigma + \tau)\overline{u}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y u}\, dy$$

for $\tilde{\eta} \in R$ where

$$R := \{\tilde{\eta} \in \mathbb{C}^M : \tilde{\eta} = \sigma + i\tau, \sigma, \tau \in \mathbb{R}^M, |\sigma| < 1/2, |\tau| < 1/2\}.$$

(II) THE FORM $t(\tilde{\eta})$ IS SECTORIAL We have

$$\begin{aligned} t(\tilde{\eta}) &= \int_Q B(D + i\sigma + \tau)u \cdot (D - i\sigma + \tau)\overline{u}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y u}\, dy \\ &= \int_Q B(D + i\sigma)u \cdot (D - i\sigma)\overline{u}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y u}\, dy - \int_Q B(\tau u) \cdot D\overline{u}\, dy \\ &\quad + \int_Q BDu \cdot (\tau\overline{u})\, dy - \int_Q B\tau u \cdot \tau\overline{u}\, dy + i \int_Q B\sigma u \cdot \tau\overline{u}\, dy + i \int_Q B\tau u \cdot \sigma\overline{u}\, dy. \end{aligned}$$

From above, it is easy to write separately the real and imaginary parts of the form $t(\tilde{\eta})$.

$$\Re t(\tilde{\eta})[u] = \int_Q B(D + i\sigma)u \cdot (D - i\sigma)\overline{u}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y u}\, dy - \int_Q B\tau u \cdot \tau\overline{u}\, dy, \tag{6.10}$$

$$\Im\, t(\tilde{\eta})[u] = \int_Q B\sigma u \cdot \tau\overline{u}\, dy + \int_Q B\tau u \cdot \sigma\overline{u}\, dy + \Im \int_Q BDu \cdot \tau\overline{u}\, dy. \tag{6.11}$$

For the real part, we can readily obtain the following estimate:

$$\Re t(\tilde{\eta})[u] + C_5 ||u||^2_{L^2_\sharp(Q)} \geq \frac{\alpha}{2}\left(||u||^2_{L^2_\sharp(Q)} + ||Du||^2_{L^2_\sharp(Q)}\right) + \delta ||\nabla_y u||^2_{L^2_\sharp(Q)}. \tag{6.12}$$

Let us define the new form $\tilde{t}(\tilde{\eta})$ by $\tilde{t}(\tilde{\eta})[u,v] = t(\tilde{\eta})[u,v] + (C_5 + C_6)(u,v)_{L^2_\sharp(Q)}$, for which it holds that

$$\Re\tilde{t}(\tilde{\eta})[u] \geq \frac{\alpha}{2}\left(||u||^2_{L^2_\sharp(Q)} + ||Du||^2_{L^2_\sharp(Q)}\right) + \delta ||\nabla_y u||^2_{L^2_\sharp(Q)} + C_6 ||u||^2_{L^2_\sharp(Q)}.$$

Also, the imaginary part of $\tilde{t}(\tilde{\eta})$ can be estimated as follows:

$$\begin{aligned}\Im\tilde{t}(\tilde{\eta})[u] &\le C_7\|u\|^2_{L^2_\sharp(Q)} + C_8\|Du\|^2_{L^2_\sharp(Q)} \\ &\overset{C_7=C_6C_9, 2C_8=\alpha C_9}{=} C_9\left(C_6\|u\|^2_{L^2_\sharp(Q)} + \frac{\alpha}{2}\|Du\|^2_{L^2_\sharp(Q)}\right) \\ &\le C_9\left(\Re\tilde{t}(\tilde{\eta})[u] - \frac{\alpha}{2}\|u\|^2_{L^2_\sharp(Q)}\right).\end{aligned}$$

This shows that $\tilde{t}(\tilde{\eta})$ is sectorial. However, sectoriality is invariant under translations by scalar multiple of identity operator in $L^2_\sharp(Q)$, therefore the form $t(\tilde{\eta})$ is also sectorial.

(iii) The form $t(\tilde{\eta})$ is closed Suppose that $u_n \xrightarrow{t} u$. This means that $u_n \to u$ in $L^2_\sharp(Q)$ and $t(\tilde{\eta})[u_n - u_m] \to 0$. As a consequence, $\Re t(\tilde{\eta})[u_n - u_m] \to 0$. By (6.12), $\|u_n - u_m\|_{H^1_\sharp(Q)} \to 0$, i.e., (u_n) is Cauchy in $H^1_\sharp(Q)$. Therefore, there exists $v \in H^1_\sharp(Q)$ such that $u_n \to v$ in $H^1_\sharp(Q)$. Due to uniqueness of limit in $L^2_\sharp(Q)$, $v = u$. Therefore, the form is closed.

(iv) The form $t(\tilde{\eta})$ is holomorphic The holomorphy of t is an easy consequence of the fact that t is a quadratic polynomial in η.

(v) 0 is an isolated eigenvalue Zero is an eigenvalue because constants belong to the kernel of $\mathcal{C}^\delta(0) = -\nabla_y \cdot (\Lambda B \Lambda^T + \delta I)\nabla_y$. We proved using Lemma 6.5 that $\mathcal{C}^\delta(0) + C_* I$ has compact resolvent. Also, C_* is an eigenvalue of $\mathcal{C}^\delta(0) + C_* I$. Therefore, C_*^{-1} is an eigenvalue of $(\mathcal{C}^\delta(0) + C_* I)^{-1}$ and C_*^{-1} is isolated. Hence, zero is an isolated point of the spectrum of $\mathcal{C}^\delta(0)$.

(vi) 0 is a geometrically simple eigenvalue Denote by $\ker \mathcal{C}^\delta(0)$ the nullspace of operator $\mathcal{C}^\delta(0)$. Let $v \in \ker \mathcal{C}^\delta(0)$, then $\int_Q (\Lambda B \Lambda^T + \delta I)\nabla_y v \cdot \nabla_y v \, dy = 0$. Due to the coercivity of the matrix $(\Lambda B \Lambda^T + \delta I)$, we obtain $\|\nabla_y v\|_{L^2_\sharp(Q)} = 0$. Hence, v is a constant. This shows that the eigenspace corresponding to eigenvalue 0 is spanned by constants, therefore, it is one-dimensional.

(vii) 0 is an algebraically simple eigenvalue Suppose that $v \in H^1_\sharp(Q)$ such that $\mathcal{C}^\delta(0)^2 v = 0$, i.e., $\mathcal{C}^\delta(0)v \in \ker \mathcal{C}^\delta(0)$. This implies that $\mathcal{C}^\delta(0)v = C$ for some generic constant C. However, by the compatibility condition for the solvability of this equation, we obtain $C = 0$. Therefore, $v \in \ker \mathcal{C}^\delta(0)$. This shows that the eigenvalue 0 is algebraically simple. □

6.5 Cell problem for quasiperiodic media

In this section, we shall recall the cell problem [OZ82] in the theory of almost periodic homogenization as well as the cell problem for the degenerate periodic operator in higher dimensions [Koz78] for quasiperiodic media.

Let e_l be the unit vector in $\mathbb{R}^d$ with 1 in the l^{th} place and 0 elsewhere. For almost periodic media, the cell problem

$$-\nabla_x \cdot (A(x)(e_l + \nabla_x w_l)) = 0 \tag{6.13}$$

is not solvable in the space of almost periodic functions. Hence, an abstract setup is required which is explained below. Let $S = \{\nabla_x \phi : \phi \in \mathrm{Trig}(\mathbb{R}^d;\mathbb{R})\}$. This is a subspace of $(B^2(\mathbb{R}^d))^d$. We shall call the closure of S in $(B^2(\mathbb{R}^d))^d$ as W. For the matrix A, we define a bilinear form on W by

$$a(w^1, w^2) = \sum_{j,k=1}^{d} \mathcal{M}(a_{jk} w^1_j w^2_k),$$

where $w^1 = (w^1_1, w^1_2, \ldots, w^1_d)$ and $w^2 = (w^2_1, w^2_2, \ldots, w^2_d)$. By coercivity of the matrix A, the bilinear form is coercive. Also, by boundedness of A, the bilinear form is continuous on $W \times W$. We also define the following linear form on W:

$$L_l(V) := -\sum_{k=1}^{d} \mathcal{M}(a_{kl}) v_k.$$

Again, by boundedness of matrix A, the linear form L_l is continuous. Hence, Lax-Milgram lemma guarantees a solution to the following problem: Find $N^l \in W$ such that $\forall\, V \in W$, we have

$$a(N^l, V) = L_l(V). \tag{6.14}$$

This is the abstract cell problem for almost periodic homogenization [OZ82] and the homogenized coefficients are defined as

$$q^*_{kl} = \mathcal{M}\left(a_{kl} + \sum_{j=1}^{d} a_{kj} N^l_j \right). \tag{6.15}$$

However, in the case of quasiperiodic media, one can also define cell problem in higher dimensions as in [Koz78]. The transformation $x \mapsto \Lambda x$ converts the cell problem in $\mathbb{R}^d$ (6.13) to a cell problem posed in Q for the degenerate periodic operator.

$$-D \cdot B(y) D\psi_l = D \cdot B(y) e_l. \tag{6.16}$$

Due to the lack of coercivity, we implement the regularizing trick as in [BLBL15]. For $0 < \delta < 1$, we seek the solution $\psi_l^\delta \in H^1_\sharp(Q)/\mathbb{R}$ to the following equation.

$$-D \cdot B(y) D\psi_l^\delta - \delta \Delta \psi_l^\delta = D \cdot B(y) e_l. \tag{6.17}$$

The solution satisfies the a priori bound $\|D\psi_l^\delta\|^2_{L^2_\sharp(Q)} + \delta \|\nabla_y \psi_l^\delta\|^2_{L^2_\sharp(Q)} \leq C$ for some generic constant C. As a consequence, $D\psi_l^\delta$ converges to some function $\chi^l \in (L^2_\sharp(Q))^d$ for a subsequence in the limit $\delta \to 0$. Using the a priori bounds, we can pass to the limit $\delta \to 0$ in the equation (6.17) to show that χ^l solves the equation (6.16) in the form

$$-D \cdot B(y) \chi^l = D \cdot B(y) e_l. \tag{6.18}$$

By elliptic regularity, $D\psi_l^\delta \in H^s_\sharp(Q)$ for all $s > 0$. As a consequence, $D\psi_l^\delta \in C^\infty(Q)$. Therefore, $\chi^l \in H^s_\sharp(Q)$ for all $s > 0$. Again, $\chi^l \in C^\infty(Q)$ and the equation (6.18) holds pointwise. Hence, we can restrict equation (6.18) to $\mathbb{R}^d$ using the matrix Λ. Define $N^l(x) = \chi^l(\Lambda x)$, then N^l solves the abstract cell problem (6.14). Therefore, the homogenized coefficients can be written in terms of the solution of the lifted cell problem (6.16).

$$q^*_{kl} = \mathcal{M}\left(a_{kl} + \sum_{j=1}^d a_{kj} N^l_j \right) = \mathcal{M}_Q \left(b_{kl} + \sum_{j=1}^d b_{kj} \chi^l_j \right). \tag{6.19}$$

Let us define the approximate homogenized tensor $A^{\delta,*} = (q^{\delta,*}_{kl})$ as

$$q^{\delta,*}_{kl} = e_k \cdot A^{\delta,*} e_l = \mathcal{M}_Q \left(b_{kl} + e_k \cdot B D\psi_l^\delta \right), \tag{6.20}$$

and the homogenized tensor $A^* = (q^*_{kl})$ of quasiperiodic media as

$$q^*_{kl} = e_k \cdot A^* e_l = \mathcal{M}_Q \left(b_{kl} + e_k \cdot B \chi^l \right). \tag{6.21}$$

then the following lemma holds true.

Lemma 6.10. *The approximate homogenized matrix* $q^{\delta,*}_{kl}$ *converges to the homogenized matrix* q^*_{kl} *of quasiperiodic media as defined in* (6.21).

Proof. The proof follows easily from the bounds that are available for ψ_l^δ. □

Remark 6.11. Another method of solving the cell problem is proposed in [WGC19] where instead of seeking solutions in the degenerate Sobolev space as in [GH16], one seeks solutions in a subspace of $H^1_\sharp(Q)$ where all the derivatives in directions orthogonal to the irrational plane are set to zero.

6.6 Characterization of homogenized tensor

Now, we shall compute derivatives with respect to η of the first regularized Bloch eigenvalue and first regularized Bloch eigenfunction at the point $\eta = 0$ and identify the homogenized tensor for quasiperiodic media. Note that the regularized Bloch eigenvalues and eigenfunctions are defined as functions of $\eta \in Y'$. The first regularized Bloch eigenfunction satisfies the following problem in Q:

$$-(D + i\eta) \cdot B(y)(D + i\eta)\phi_1^\delta(y;\eta) - \delta\Delta\phi_1^\delta(y;\eta) = \lambda_1^\delta(\eta)\phi_1^\delta(y;\eta). \tag{6.22}$$

We know that $\lambda_1^\delta(0) = 0$. For $\eta \in Y'$, recall that $\mathcal{C}^\delta(\eta) = -(D+i\eta)\cdot B(y)(D+i\eta)-\delta\Delta$. In the rest of this section, we will suppress the dependence on y for convenience. For $l = 1, 2, \ldots, d$, differentiate equation (6.22) with respect to η_l to obtain

$$\frac{\partial \mathcal{C}^\delta}{\partial \eta_l}(\eta)\phi_1^\delta(\eta) + \mathcal{C}^\delta(\eta)\frac{\partial \phi_1^\delta}{\partial \eta_l}(\eta) = \lambda_1^\delta(\eta)\frac{\partial \phi_1^\delta}{\partial \eta_l}(\eta) + \frac{\partial \lambda_1^\delta}{\partial \eta_l}(\eta)\phi_1^\delta(\eta), \tag{6.23}$$

where $\dfrac{\partial \mathcal{C}}{\partial \eta_l}(\eta) = -iD\cdot(Be_l) - ie_l\cdot(BD) + e_l\cdot B\eta + \eta\cdot Be_l$, where e_l is the unit vector in $\mathbb{R}^d$ with 1 in the l^{th} place and 0 elsewhere. We multiply (6.23) by $\overline{\phi_1^\delta(\eta)}$, take mean value over Q and set $\eta = 0$ to get $\dfrac{\partial \lambda_1^\delta}{\partial \eta_l}(0) = 0$ for all $l = 1, 2, \ldots, d$.

On the other hand, if we set $\eta = 0$ in (6.23), we obtain

$$\mathcal{C}^\delta(0)\frac{\partial \phi_1^\delta}{\partial \eta_l}(0) = -\frac{\partial \mathcal{C}^\delta}{\partial \eta_l}(0)\phi_1^\delta(0),$$

or

$$(-D\cdot B(y)D - \delta\Delta)\,\frac{\partial \phi_1^\delta}{\partial \eta_l}(0) = D\cdot B(y)e_l i\phi_1^\delta(0).$$

Hence, $\psi_l^\delta - \dfrac{1}{i\phi^\delta{}_1(0)}\dfrac{\partial \phi_1^\delta}{\partial \eta_l}(0)$ is a constant.

Now, differentiate (6.23) with respect to η_k to obtain

$$\left(\frac{\partial^2 \mathcal{C}^\delta}{\partial\eta_k\partial\eta_l}(\eta) - \frac{\partial^2 \lambda_1^\delta}{\partial\eta_k\partial\eta_l}(\eta)\right)\phi_1^\delta(\eta) + \left(\frac{\partial \mathcal{C}^\delta}{\partial\eta_k}(\eta) - \frac{\partial \lambda_1^\delta}{\partial\eta_k}(\eta)\right)\frac{\partial \phi_1^\delta}{\partial \eta_l}(\eta) +$$
$$\left(\frac{\partial \mathcal{C}^\delta}{\partial\eta_l}(\eta) - \frac{\partial \lambda_1^\delta}{\partial\eta_l}(\eta)\right)\frac{\partial \phi_1^\delta}{\partial \eta_k}(\eta) + \left(\mathcal{C}^\delta(\eta) - \lambda_1^\delta(\eta)\right)\frac{\partial^2 \phi_1^\delta}{\partial\eta_l\partial\eta_k}(\eta) = 0. \tag{6.24}$$

Multiply with $\overline{\phi_1^\delta(\eta)}$, take mean value over Q and set $\eta = 0$ to obtain

$$\frac{1}{2}\frac{\partial^2 \lambda_1^\delta}{\partial\eta_k\partial\eta_l}(0) = \mathcal{M}_Q\left(b_{kl} + \frac{1}{2}e_k\cdot BD\psi_l^\delta + \frac{1}{2}e_l\cdot BD\psi_k^\delta\right). \tag{6.25}$$

Thus, we have proved the following theorem:

Theorem 6.12. *The regularized first Bloch eigenvalue and eigenfunction satisfy:*

1. $\lambda_1^\delta(0) = 0$.

2. *The eigenvalue* $\lambda_1^\delta(\eta)$ *has a critical point at* $\eta = 0$, *i.e.,*

$$\frac{\partial \lambda_1^\delta}{\partial \eta_l}(0) = 0, \forall l = 1, 2, \ldots, d. \tag{6.26}$$

3. *For* $l = 1, 2, \ldots, d$, *the derivative of the eigenvector* $(\partial \phi_1^\delta / \partial \eta_l)(0)$ *satisfies:* $(\partial \phi_1^\delta / \partial \eta_l)(y; 0) - i\phi_1^\delta(y; 0)\psi_l^\delta(y)$ *is a constant in* y *where* ψ_l^δ *solves the cell problem* (6.17).

4. *The Hessian of the first Bloch eigenvalue at* $\eta = 0$ *is twice the approximate homogenized matrix* $q_{kl}^{\delta,*}$ *as defined in* (6.20), *i.e.,*

$$\frac{1}{2} \frac{\partial^2 \lambda_1^\delta}{\partial \eta_k \partial \eta_l}(0) = q_{kl}^{\delta,*} \tag{6.27}$$

□

6.7 Quasiperiodic Bloch transform

We shall normalize $\phi_1^\delta(y; 0)$ to be $(2\pi)^{-d/2}$. The Bloch problem at ϵ-scale is given by

$$-(D_{y'} + i\xi) \cdot B(y'/\epsilon)(D_{y'} + i\xi)\phi_1^{\delta,\epsilon}(y'; \xi) - \delta \Delta_{y'} \phi_1^{\delta,\epsilon}(y'; \xi) = \lambda_1^{\delta,\epsilon}(\xi)\phi_1^{\delta,\epsilon}(y'; \xi) \tag{6.28}$$

for $y \in \epsilon Q$ and $\xi \in \epsilon Y'$. Due to the transformation $y = y'/\epsilon$ and $\eta = \epsilon\xi$, we have $\lambda_1^{\delta,\epsilon}(\xi) = \epsilon^{-2}\lambda_1^\delta(\epsilon\xi)$ and $\phi_1^{\delta,\epsilon}(y'; \xi) = \phi_1^\delta(y'/\epsilon; \epsilon\xi)$. The above equation holds pointwise for $y' \in \epsilon Q$ and is analytic for $\xi \in \epsilon^{-1}U^\delta$. For the purpose of Bloch wave homogenization, we need to restrict the regularized Bloch eigenvalues and eigenfunctions to $\mathbb{R}^d$ using the matrix Λ. Let us define $\tilde{\phi}_1^{\delta,\epsilon}(x; \xi) := \phi_1^\delta(\frac{\Lambda x}{\epsilon}; \epsilon\xi)$. Also define $\beta_1^{\delta;\epsilon}(y', \xi) := \sqrt{\delta}\Delta_{y'}\phi_1^{\delta,\epsilon}(y'; \xi)$ and its restriction $\tilde{\beta}_1^{\delta;\epsilon}(x, \xi) = \sqrt{\delta}\Delta_x \phi_1^{\delta,\epsilon}(\Lambda x; \xi)$, then the restriction of the first regularized Bloch eigenfunction satisfies the following approximate spectral problem in $\mathbb{R}^d$.

$$-(\nabla_x + i\xi) \cdot A\left(\frac{x}{\epsilon}\right)(\nabla_x + i\xi)\tilde{\phi}_1^{\delta,\epsilon}(x; \xi) - \sqrt{\delta}\tilde{\beta}_1^{\delta;\epsilon}(x, \xi) = \lambda_1^{\delta,\epsilon}(\xi)\tilde{\phi}_1^{\delta,\epsilon}(x; \xi). \tag{6.29}$$

We can compare this to our original goal of solving equation (6.4) in $\mathbb{R}^d$. Although we could not solve the exact quasiperiodic Bloch spectral problem, we could solve an approximate quasiperiodic Bloch problem using the lifted periodic problem. Interestingly, the functions $\tilde{\phi}_1^{\delta,\epsilon}(x;\xi)$ and $\tilde{\beta}_1^{\delta;\epsilon}(x,\xi)$ are quasiperiodic functions of the first variable.

Now we can define a dominant Bloch coefficient for compactly supported functions in $\mathbb{R}^d$ by employing the first regularized Bloch eigenfunction as follows: Let $g \in H^{-1}(\mathbb{R}^d)$ with compact support, then define

$$\mathcal{B}_1^{\delta,\epsilon} g(\xi) := \left\langle g(x), e^{-ix\cdot\xi}\,\overline{\tilde{\phi}_1^{\delta,\epsilon}(x;\xi)}\right\rangle_{H^{-1},H^1}. \tag{6.30}$$

For the next section, we need to know the limit of Bloch transform of a sequence of functions as below.

Theorem 6.13. *Let $K \subseteq \mathbb{R}^d$ be a compact set and (g^ϵ) be a sequence of functions in $L^2(\mathbb{R}^d)$ such that $g^\epsilon = 0$ outside K. Suppose that $g^\epsilon \rightharpoonup g$ in $L^2(\mathbb{R}^d)$-weak for some function $g \in L^2(\mathbb{R}^d)$. Then it holds that*

$$\chi_{\epsilon^{-1}U^\delta}\mathcal{B}_1^{\delta,\epsilon} g^\epsilon \rightharpoonup \widehat{g}$$

in $L^2_{loc}(\mathbb{R}^d_\xi)$-weak, where $\widehat{g}$ denotes the Fourier transform of g.

Proof. The function $\mathcal{B}_1^{\delta,\epsilon} g^\epsilon$ is defined for $\xi \in \epsilon^{-1}Y'$. However, we shall treat it as a function on $\mathbb{R}^d$ by extending it outside $\epsilon^{-1}U^\delta$ by zero. We can write

$$\mathcal{B}_1^{\delta,\epsilon} g^\epsilon(\xi) = \int_{\mathbb{R}^d} g(x)e^{-ix\cdot\xi}\overline{\tilde{\phi}_1^{\delta,\epsilon}}(x;0)\,dx + \int_{\mathbb{R}^d} g(x)e^{-ix\cdot\xi}\left(\tilde{\phi}_1^{\delta}\left(\frac{x}{\epsilon};\epsilon\xi\right) - \tilde{\phi}_1^{\delta}\left(\frac{x}{\epsilon};0\right)\right)dx.$$

The first term above converges to the Fourier transform of g on account of the normalization of $\phi_1(y;0)$ whereas the second term goes to zero since it is $O(\epsilon\xi)$ due to the Lipschitz continuity of the first regularized Bloch eigenfunction. More details including the proof of Lipschitz continuity of Bloch eigenvalues and eigenfunctions may be found in [CV97]. □

6.8 Homogenization theorem

In this section, we shall prove the following homogenization theorem for quasiperiodic media and prove it using the Bloch wave method. We shall assume summation over repeated indices for ease of notation.

Theorem 6.14. *Let Ω be an open set in $\mathbb{R}^d$ and $f \in L^2(\Omega)$. Let $u^\epsilon \in H^1(\Omega)$ be such that u^ϵ converges weakly to u^* in $H^1_0(\Omega)$, and*

$$\mathcal{A}^\epsilon u^\epsilon = f \text{ in } \Omega. \tag{6.31}$$

Then

1. *For all $k = 1, 2, \ldots, d$, we have the following convergence of fluxes:*
$$a^\epsilon_{kl}(x)\frac{\partial u^\epsilon}{\partial x_l}(x) \rightharpoonup q^*_{kl}\frac{\partial u^*}{\partial x_l}(x) \text{ in } L^2(\Omega)\text{-weak.} \tag{6.32}$$

2. *The limit u^* satisfies the homogenized equation:*
$$\mathcal{A}^{\text{hom}}u^* = -\frac{\partial}{\partial x_k}\left(q^*_{kl}\frac{\partial u^*}{\partial x_l}\right) = f \text{ in } \Omega. \tag{6.33}$$

The proof of Theorem 6.14 is divided into the following steps. We begin by localizing the equation (6.31) which is posed on Ω, so that it is posed on $\mathbb{R}^d$. We take the quasiperiodic Bloch transform $\mathcal{B}^{\delta,\epsilon}_1$ of this equation and pass to the limit $\epsilon \to 0$, followed by the limit $\delta \to 0$.

STEP 1: Let ψ_0 be a fixed smooth function supported in a compact set $K \subset \mathbb{R}^d$. Since u^ϵ satisfies $\mathcal{A}^\epsilon u^\epsilon = f$, $\psi_0 u^\epsilon$ satisfies

$$\mathcal{A}^\epsilon(\psi_0 u^\epsilon)(x) = \psi_0 f(x) + g^\epsilon(x) + h^\epsilon(x) \text{ in } \mathbb{R}^d, \tag{6.34}$$

where

$$g^\epsilon(x) := -\frac{\partial \psi_0}{\partial x_k}(x)a^\epsilon_{kl}(x)\frac{\partial u^\epsilon}{\partial x_l}(x), \tag{6.35}$$
$$h^\epsilon(x) := -\frac{\partial}{\partial x_k}\left(\frac{\partial \psi_0}{\partial x_l}(x)a^\epsilon_{kl}(x)u^\epsilon(x)\right), \tag{6.36}$$

STEP 2: Taking the first Bloch transform of both sides of the equation (6.34), we obtain for $\xi \in \epsilon^{-1}U^\delta$ a.e.

$$\mathcal{B}^{\delta,\epsilon}_1(\mathcal{A}^\epsilon(\psi_0 u^\epsilon))(\xi) = \mathcal{B}^{\delta,\epsilon}_1(\psi_0 f)(\xi) + \mathcal{B}^{\delta,\epsilon}_1 g^\epsilon(\xi) + \mathcal{B}^{\delta,\epsilon}_1 h^\epsilon(\xi). \tag{6.37}$$

STEP 3: Observe that $\psi_0 u^\epsilon \in H^1(\mathbb{R}^d)$. We have

$$\begin{aligned}
\mathcal{B}^{\delta,\epsilon}_1(\mathcal{A}^\epsilon(\psi_0 u^\epsilon)) &= \int_{\mathbb{R}^d} A(x/\epsilon)\nabla(\psi_0 u^\epsilon)(x)\cdot\nabla(e^{-ix\cdot\xi}\overline{\tilde{\phi}^{\delta,\epsilon}_1}(x;\xi))\,dx \\
&= \lambda^{\delta,\epsilon}_1(\xi)\int_{\mathbb{R}^d}(\psi_0 u^\epsilon)(x)e^{-ix\cdot\xi}\overline{\tilde{\phi}^\epsilon_1}(x;\xi)\,dx + \sqrt{\delta}\int_{\mathbb{R}^d}(\psi_0 u^\epsilon)(x)e^{-ix\cdot\xi}\overline{\tilde{\beta}^{\delta,\epsilon}_1}(x;\xi)\,dx \\
&= \lambda^{\delta,\epsilon}_1(\xi)\mathcal{B}^\epsilon_1(\psi_0 u^\epsilon) + \sqrt{\delta}\int_{\mathbb{R}^d}(\psi_0 u^\epsilon)(x)e^{-ix\cdot\xi}\overline{\tilde{\beta}^{\delta,\epsilon}_1}(x;\xi)\,dx
\end{aligned} \tag{6.38}$$

Step 4: In this step, we shall obtain bounds for $\tilde{\beta}_1^{\delta,\epsilon}$. This is done by employing the analyticity of the first regularized Bloch eigenfunction in a neighborhood of $\eta = 0$. Let us write

$$\phi_1^\delta(y;\eta) = \phi_1^\delta(y;0) + \eta_l \frac{\partial \phi_1^\delta}{\partial \eta_l}(y;0) + \gamma^\delta(y;\eta),$$

where $\gamma^\delta(y;0) = 0$, $\frac{\partial \gamma^\delta}{\partial \eta_l}(y;0) = 0$ and $\sqrt{\delta}\gamma^\delta(\cdot;\eta) = O(|\eta|^2)$ in $L^\infty(U^\delta; H^1_\sharp(Q))$ where the order is uniform in δ. Therefore, $\sqrt{\delta}\dfrac{\partial^2\gamma^\delta}{\partial y_k^2}(\cdot;\eta) = O(|\eta|^2)$ in $L^\infty(U^\delta; H^{-1}_\sharp(Q))$ where the order is uniform in δ. Now,

$$\phi_1^{\delta,\epsilon}(y';\xi) = \phi_1^\delta\left(\frac{y'}{\epsilon};\epsilon\xi\right) = \phi_1^\delta\left(\frac{y'}{\epsilon};0\right) + \epsilon\xi_l \frac{\partial \phi_1^\delta}{\partial \eta_l}\left(\frac{y'}{\epsilon};0\right) + \gamma^\delta\left(\frac{y'}{\epsilon};\epsilon\xi\right). \tag{6.39}$$

Let us define $\alpha_l^{\delta,\epsilon}(y') := \dfrac{\epsilon}{i\phi_1^\delta(y'/\epsilon;0)}\dfrac{\partial\phi_1^\delta}{\partial\eta_l}\left(\dfrac{y'}{\epsilon};0\right)$, then $\alpha_l^{\delta,\epsilon}(y') \in H^1_\sharp(\epsilon Q)$ solves the cell problem at ϵ-scale posed in ϵQ, i.e.,

$$-D_{y'} \cdot B^\epsilon(y')D_{y'}\alpha_l^{\delta,\epsilon} - \delta\Delta_{y'}\alpha_l^{\delta,\epsilon} = D_{y'} \cdot B^\epsilon(y')e_l, \tag{6.40}$$

which provides the estimate

$$||D_{y'}\alpha_l^{\delta,\epsilon}||^2_{L^2_\sharp(\epsilon Q)} + \delta||\nabla_{y'}\alpha_l^{\delta,\epsilon}||^2_{L^2_\sharp(\epsilon Q)} \leq C, \tag{6.41}$$

for some generic constant C not depending on ϵ and δ. Therefore, we get

$$\left(\sqrt{\delta}\Delta_{y'}\alpha_l^{\delta,\epsilon}\right) \text{is bounded uniformly in } H^{-1}_\sharp(\epsilon Q). \tag{6.42}$$

Differentiating the equation (6.39) with respect to y' twice, we obtain

$$\frac{\partial^2\phi_1^{\delta,\epsilon}}{\partial y_k'^2}(y',\xi) = \xi_l\epsilon\frac{\partial^2}{\partial y_k'^2}\frac{\partial\phi_1^{\delta,\epsilon}}{\partial\eta_l}\left(y';0\right) + \epsilon^{-2}\frac{\partial^2\gamma^\delta}{\partial y_k^2}\left(\frac{y'}{\epsilon};\epsilon\xi\right).$$

For ξ belonging to the set $\{\xi : \epsilon\xi \in U^\delta \text{ and } |\xi| \leq M\}$, we have

$$\sqrt{\delta}\left|\frac{\partial^2\gamma^\delta}{\partial y_k^2}(\cdot;\eta)\right| \leq C\epsilon^2 M^2.$$

$$\text{Therefore, } \left(\sqrt{\delta}\epsilon^{-2}\frac{\partial^2\gamma^\delta}{\partial y_k^2}(y'/\epsilon;\epsilon\xi)\right) \text{ is bounded uniformly in } L^2_{loc}(\mathbb{R}^d_\xi; H^{-1}_\sharp(\epsilon Q)). \tag{6.43}$$

From (6.42) and (6.43), we have $\beta_1^{\delta,\epsilon}(y',\xi) = \sqrt{\delta}\xi_l i\phi_1^\delta\left(\frac{y'}{\epsilon};0\right)\Delta_{y'}\alpha_l^{\delta,\epsilon} + \frac{\sqrt{\delta}}{\epsilon^2}\sum_{k=1}^M \frac{\partial^2\gamma^\delta}{\partial y_k^2}\left(\frac{y'}{\epsilon};\epsilon\xi\right)$ is bounded uniformly in $L^2_{loc}(\mathbb{R}^d_\xi; H^{-1}_\sharp(\epsilon Q))$. As a consequence,we obtain $\left(\tilde{\beta}_1^{\delta,\epsilon}\right)$ is bounded uniformly in $L^2_{loc}(\mathbb{R}^d_\xi; H^{-1}_{loc}(\mathbb{R}^d))$.

STEP 5: Now, we are ready to pass to the limit $\epsilon \to 0$ in the equation (6.37). In view of equation (6.38), equation (6.37) becomes

$$\lambda_1^{\delta,\epsilon}(\xi)\mathcal{B}_1^{\epsilon}(\psi_0 u^{\epsilon}) + \sqrt{\delta}\int_{\mathbb{R}^d}(\psi_0 u^{\epsilon})(x)e^{-ix\cdot\xi}\overline{\tilde{\beta}_1^{\delta,\epsilon}}(x;\xi)\,dx = \mathcal{B}_1^{\delta,\epsilon}(\psi_0 f)(\xi) + \mathcal{B}_1^{\delta,\epsilon}g^{\epsilon}(\xi) + \mathcal{B}_1^{\delta,\epsilon}h^{\epsilon}(\xi). \tag{6.44}$$

Let us denote $\Upsilon^{\delta,\epsilon}(\xi) = \int_{\mathbb{R}^d}(\psi_0 u^{\epsilon})(x)e^{-ix\cdot\xi}\overline{\tilde{\beta}_1^{\delta,\epsilon}}(x;\xi)\,dx$. Let K_2 be a compact subset of $\mathbb{R}^d_{\xi}$. From the previous step, we have

$$||\Upsilon^{\delta,\epsilon}||_{L^2(K_2)} \lesssim ||\tilde{\beta}_1^{\delta,\epsilon}||_{L^2(K_2;H^{-1}(K))}$$

Hence, $\Upsilon^{\delta,\epsilon}$ is bounded in $L^2_{loc}(\mathbb{R}^d_{\xi})$ independent of δ and ϵ. Therefore, it converges weakly to Υ^{δ} in $L^2_{loc}(\mathbb{R}^d_{\xi})$ for a subsequence. Once more, since the sequence $\Upsilon^{\delta,\epsilon}$ is bounded uniformly in δ, the weak limit Υ^{δ} is also bounded uniformly in δ.

The proofs of convergences of all terms except the second term on LHS in (6.44) follows the same lines as in [CV97]. Therefore, passing to the limit in (6.44) as $\epsilon \to 0$ we obtain for $\xi \in \mathbb{R}^d$

$$\frac{1}{2}\frac{\partial^2\lambda_1^{\delta}}{\partial\eta_k\partial\eta_l}(0)\xi_k\xi_l\widehat{\psi_o u^*}(\xi) + \sqrt{\delta}\Upsilon^{\delta}(\xi) = \widehat{(\psi_0 f)}(\xi) - \widehat{\left(\frac{\partial\psi_0}{\partial x_k}(x)\sigma_k^*(x)\right)}(\xi) - i\xi_k q^*_{kl}\widehat{\left(\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\right)}(\xi), \tag{6.45}$$

where σ_k^* is the weak limit of the flux $a^{\epsilon}_{kl}(x)\frac{\partial u^{\epsilon}}{\partial x_l}(x)$.

STEP 6: Now, we may pass to the limit in equation (6.45) as $\delta \to 0$. Using Theorem 6.12, Lemma 6.10, and the uniform in δ bound for Υ^{δ}, we obtain the following equation.

$$q^*_{kl}\xi_k\xi_l\widehat{\psi_0 u^*}(\xi) = \widehat{(\psi_0 f)}(\xi) - \widehat{\left(\frac{\partial\psi_0}{\partial x_k}(x)\sigma_k^*(x)\right)}(\xi) - i\xi_k q^*_{kl}\widehat{\left(\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\right)}(\xi), \tag{6.46}$$

where σ_k^* is the weak limit of the flux $a^{\epsilon}_{kl}(x)\frac{\partial u^{\epsilon}}{\partial x_l}(x)$.

The rest of the steps involving the identification of σ_k^* and the homogenized equation are the same as in [CV97] and are therefore omitted.

6.9 Comments

This chapter ties in thematically with the rest of the thesis. As in previous chapters, it was difficult to directly analyze the original operator and hence an approximation was employed. The kind of bounds that appear are reminiscent of the method of vanishing viscosity in conservation laws.

The problem also seems to be amenable to a different kind of analysis. Instead of solving the regularized Bloch problem, one could create a subspace of H^1 periodic functions that only live on the "hyperplane" defined by the matrix Λ, i.e., it is constant along the orthogonal directions. One could do a Bloch decomposition of the degenerate operator in this subspace. Indeed, the cell problem too could be posed in this space. However, it is not clear whether its solution matches up with that of the standard cell problem. The subspace appears to be too small for homogenization.

Duerinckx et. al. [DGS18] argue that the spectrum of the shifted operator corresponding to a degenerate operator or a quasiperiodic operator is pure point and dense. We are however able to achieve a discrete spectrum through regularization of the operator. This turns out to be sufficient for the purposes of homogenization.

Appendix A

Perturbation Theory of holomorphic family of type (B)

In this section, we show that a perturbation in the coefficients of the operator $\mathcal{A}$ gives rise to a corresponding holomorphic family of sectorial forms of type (a). Further, the selfadjointness of the forms coupled with the compactness of the resolvent for the operator family ensures that it is a selfadjoint holomorphic family of type (B). For definition of these notions, see Kato [Kat95].

Let $A \in M_B^{>}$ and $B = (b_{kl})$ be a symmetric matrix with $L^\infty_\sharp(Y, \mathbb{R})$ entries. Then for $\sigma < \frac{\alpha}{\|B\|_{L^\infty}}$, $A + \sigma B$ belongs to $M_B^{>}$, where α is a coercivity constant for A, as in ((A3)). For a fixed $\eta_0 \in Y'$ and for $\sigma_0 := \frac{\alpha}{2\|B\|_{L^\infty}}$, let us define the operator family

$$\mathcal{A}(\eta_0)(\tau) = -(\nabla + i\eta_0) \cdot (A + \tau B)(\nabla + i\eta_0), \quad \tau \in R,$$

where $R = \{z \in \mathbb{C} : |\mathrm{Re}(z)| < \sigma_0, |\mathrm{Im}(z)| < \sigma_0\}$. For real τ, $-\sigma_0 < \tau < \sigma_0$, $A + \tau B$ is coercive with a coercivity constant $\alpha/2$. The holomorphic family of sesquilinear forms $\mathfrak{t}(\tau)$ associated to operator $\mathcal{A} + \tau\mathcal{B}$, with the τ-independent domain $\mathfrak{D}(\mathfrak{t}(\tau)) = H^1_\sharp(Y)$, is defined as

$$\begin{aligned}\mathfrak{t}(\tau)[u, v] := \int_Y (a_{kl}(y) + \tau b_{kl}(y)) \frac{\partial u}{\partial y_l} \frac{\partial \overline{v}}{\partial y_k}\, dy + i\eta_{0,l} \int_Y (a_{kl}(y) + \tau b_{kl}(y))\, u \frac{\partial \overline{v}}{\partial y_k}\, dy \\ - i\eta_{0,k} \int_Y (a_{kl}(y) + \tau b_{kl}(y))\, \overline{v} \frac{\partial u}{\partial y_l}\, dy + \eta_{0,l}\eta_{0,k} \int_Y (a_{kl}(y) + \tau b_{kl}(y))\, u\overline{v}\, dy,\end{aligned}$$

where $\eta_0 := (\eta_{0,1}, \eta_{0,2}, \ldots, \eta_{0,d})$ and summation over repeated indices is assumed.

Theorem A.1. $\mathfrak{t}(\tau)$ *is a holomorphic family of type* (a).

Proof. The quadratic form associated with $\mathfrak{t}(\tau)$ is as follows:

$$\begin{aligned}\mathfrak{t}(\tau)[u] := &\int_Y (a_{kl}(y) + \tau b_{kl}(y)) \frac{\partial u}{\partial y_l}\frac{\partial \overline{u}}{\partial y_k}\, dy + i\eta_{0,l}\int_Y (a_{kl}(y) + \tau b_{kl}(y))\, u\frac{\partial \overline{u}}{\partial y_k}\, dy \\ &- i\eta_{0,k}\int_Y (a_{kl}(y) + \tau b_{kl}(y))\, \overline{u}\frac{\partial u}{\partial y_l}\, dy + \eta_{0,k}\eta_{0,l}\int_Y (a_{kl}(y) + \tau b_{kl}(y))\, u\overline{u}\, dy.\end{aligned}$$

(i) $\mathfrak{t}(\tau)$ is sectorial.

Let us write $\tau = \rho + i\gamma$, then the quadratic form $\mathfrak{t}(\tau)$ can be written as the sum of its real and imaginary parts:

$$\mathfrak{t}(\tau) = \Re\mathfrak{t}(\tau)[u] + i\,\Im\mathfrak{t}(\tau)[u]$$

where the real part is

$$\begin{aligned}\Re\mathfrak{t}(\tau)[u] := &\int_Y (a_{kl}(y) + \rho b_{kl}(y)) \frac{\partial u}{\partial y_l}\frac{\partial \overline{u}}{\partial y_k}\, dy + i\eta_{0,l}\int_Y (a_{kl}(y) + \rho b_{kl}(y))\, u\frac{\partial \overline{u}}{\partial y_k}\, dy \\ &- i\eta_{0,k}\int_Y (a_{kl}(y) + \rho b_{kl}(y))\, \overline{u}\frac{\partial u}{\partial y_l}\, dy + \eta_{0,k}\eta_{0,l}\int_Y (a_{kl}(y) + \rho b_{kl}(y))\, u\overline{u}\, dy,\end{aligned} \tag{A.1}$$

and the imaginary part is

$$\begin{aligned}\Im\mathfrak{t}(\tau)[u] := &\int_Y \gamma b_{kl}(y)\frac{\partial u}{\partial y_l}\frac{\partial \overline{u}}{\partial y_k}\, dy - 2\,\mathrm{Im}\left(\eta_{0,l}\int_Y \gamma b_{kl}(y) u\frac{\partial \overline{u}}{\partial y_k}\, dy\right) \\ &+ \eta_{0,k}\eta_{0,l}\int_Y \gamma b_{kl}(y) u\overline{u}\, dy.\end{aligned} \tag{A.2}$$

The real part (A.1) of $\mathfrak{t}(\tau)[u]$ may also be written as

$$\begin{aligned}\mathrm{Re}\,\mathfrak{t}(\tau)[u] := &\int_Y (a_{kl}(y) + \rho b_{kl}(y)) \frac{\partial u}{\partial y_l}\frac{\partial \overline{u}}{\partial y_k}\, dy \\ &+ 2\,\mathrm{Re}\left(i\eta_{0,l}\int_Y (a_{kl}(y) + \rho b_{kl}(y))\, u\frac{\partial \overline{u}}{\partial y_k}\, dy\right) \\ &+ \eta_{0,k}\eta_{0,l}\int_Y (a_{kl}(y) + \rho b_{kl}(y))\, u\overline{u}\, dy.\end{aligned} \tag{A.3}$$

The first term in (A.3) is estimated from below as follows:

$$\int_Y (a_{kl}(y) + \rho b_{kl}(y)) \frac{\partial u}{\partial y_l}\frac{\partial \overline{u}}{\partial y_k}\, dy \geq \frac{\alpha}{2}\int_Y |\nabla u|^2\, dy. \tag{A.4}$$

The second term in (A.3) may be bounded from above as follows:

$$\begin{aligned}\mathrm{Re}\left(i\eta_{0,l}\int_Y (a_{kl}(y) + \rho b_{kl}(y))\, u\frac{\partial \overline{u}}{\partial y_k}\, dy\right) &\leq C_1||u||_{L^2_\sharp(Y)}||\nabla u||_{L^2_\sharp(Y)} \\ &\leq C_1 C_*||u||^2_{L^2_\sharp(Y)} + \frac{C_1}{C_*}||\nabla u||^2_{L^2_\sharp(Y)} \\ &= C_2||u||^2_{L^2_\sharp(Y)} + \frac{\alpha}{4}||\nabla u||^2_{L^2_\sharp(Y)},\end{aligned} \tag{A.5}$$

where $C_* = \frac{4C_1}{\alpha}$ and C_1, C_2 are some constants independent of u and ρ.

The last term in (A.3) is estimated as

$$\eta_{0,k}\eta_{0,l}\int_Y (a_{kl}(y) + \rho b_{kl}(y))\, u\bar{u}\ dy \leq C_3\|u\|^2_{L^2_\sharp(Y)}, \tag{A.6}$$

for some $C_3 > 0$.

Finally, combining (A.4), (A.5) and (A.6), we obtain

$$\Re\mathfrak{t}(\tau)[u] \geq \frac{\alpha}{4}\|u\|^2_{H^1_\sharp(Y)} - C_4\|u\|^2_{L^2_\sharp(Y)}, \tag{A.7}$$

for some $C_4 > 0$.

Estimating the imaginary part (A.2) from above, we obtain

$$|\Im\mathfrak{t}(\tau)[u]| \leq C_5\|\nabla u\|^2_{L^2_\sharp(Y)} + C_6\|u\|^2_{L^2_\sharp(Y)}, \tag{A.8}$$

for some positive C_5, C_6.

Now, choose a scalar C_7 so that $C_7 = \frac{4C_5}{\alpha}$.

The inequality (A.7) may be written as

$$\Re\mathfrak{t}(\tau)[u] + C_4\|u\|^2_{L^2_\sharp(Y)} + \frac{C_6}{C_7}\|u\|^2_{L^2_\sharp(Y)} \geq \frac{\alpha}{4}\|u\|^2_{H^1_\sharp(Y)} + \frac{C_6}{C_7}\|u\|^2_{L^2_\sharp(Y)}. \tag{A.9}$$

Now, we define a new quadratic form $\tilde{\mathfrak{t}}[u] := \mathfrak{t}[u] + (C_4 + \frac{C_6}{C_7})\|u\|^2_{L^2_\sharp Y}$, then inequality (A.9) becomes

$$\Re\tilde{\mathfrak{t}}(\tau)[u] \geq \frac{\alpha}{4}\|u\|^2_{H^1_\sharp(Y)} + \frac{C_6}{C_7}\|u\|^2_{L^2_\sharp(Y)}. \tag{A.10}$$

This may be further written as

$$\Re\tilde{\mathfrak{t}}(\tau)[u] - \frac{\alpha}{4}\|u\|^2_{L^2_\sharp(Y)} \geq \frac{\alpha}{4}\|\nabla u\|^2_{L^2_\sharp(Y)} + \frac{C_6}{C_7}\|u\|^2_{L^2_\sharp(Y)}. \tag{A.11}$$

On multiplying throughout by C_7, the inequality (A.11) becomes

$$C_7\left\{\Re\tilde{\mathfrak{t}}(\tau)[u] - \frac{\alpha}{4}\|u\|^2_{L^2_\sharp(Y)}\right\} \geq C_5\|\nabla u\|^2_{L^2_\sharp(Y)} + C_6\|u\|^2_{L^2_\sharp(Y)}. \tag{A.12}$$

Since $\Im\tilde{\mathfrak{t}}(\tau)[u] = \Im\mathfrak{t}(\tau)[u]$, combining the inequalities (A.8) and (A.12), we obtain

$$|\Im\tilde{\mathfrak{t}}(\tau)[u]| \leq C_7\left\{\Re\tilde{\mathfrak{t}}(\tau)[u] - \frac{\alpha}{4}\|u\|^2_{L^2_\sharp(Y)}\right\}.$$

This proves that the form $\tilde{\mathfrak{t}}(\tau)$ is sectorial. However, the property of sectoriality is invariant under a shift. Therefore, $\mathfrak{t}(\tau)$ is sectorial, as well.

(ii) $\mathfrak{t}(\tau)$ is closed.

This follows from the inequality (A.10). If $u_n \xrightarrow{\mathfrak{t}-\text{convergence}} u$, then $\Re\mathfrak{t}(\tau)[u_n - u_m] \to 0$ as $n, m \to \infty$. By (A.10), (u_n) is a Cauchy sequence in $H^1_\sharp(Y)$. By completeness, there is $v \in H^1_\sharp(Y)$ to which the sequence converges. However, $\mathfrak{t}$-convergence implies L^2 convergence, and therefore, $u = v$. Clearly, $\mathfrak{t}(\tau)[u_n - u] \to 0$.

(iii) $\mathfrak{t}(\tau)$ is a holomorphic family of type (a).

We have proved that $\mathfrak{t}(\tau)[u]$ is sectorial and closed. It remains to prove that the form is holomorphic. This is easily done since $\mathfrak{t}(\tau)[u]$ is linear in τ for each fixed $u \in H^1_\sharp(Y)$. □

The first representation theorem of Kato ensures that there exists a unique m-sectorial operator with domain contained in $H^1_\sharp(Y)$ associated with each $\mathfrak{t}(\tau)$. A proof may be found in [Kat95, p.322]. The family of such operators associated with a holomorphic family of sesquilinear forms of type (a) is called a holomorphic family of type (B). The aforementioned m-sectorial operator is given by

$$\mathcal{A}(\eta_0)(\tau) = -(\nabla + i\eta_0) \cdot (A + \tau B)(\nabla + i\eta_0).$$

It follows from the symmetry of the matrix $A + \tau B$ that the family $\mathcal{A}(\eta_0)(\tau)$ is a selfadjoint holomorphic family of type (B). Moreover, by the compact embedding of $H^1_\sharp(Y)$ in $L^2_\sharp(Y)$, the operator $\mathcal{A}(\eta_0)(\tau) + C^* I$ has compact resolvent for each $\tau \in R$ for some appropriate constant C^*, independent of $\tau \in R$.

Hence, by Kato-Rellich Theorem, there exists a complete orthonormal set of eigenvectors associated with the operator family $\mathcal{A}(\eta_0)(\tau)$ which are analytic for the whole interval $-\sigma_0 < \tau < \sigma_0$.

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