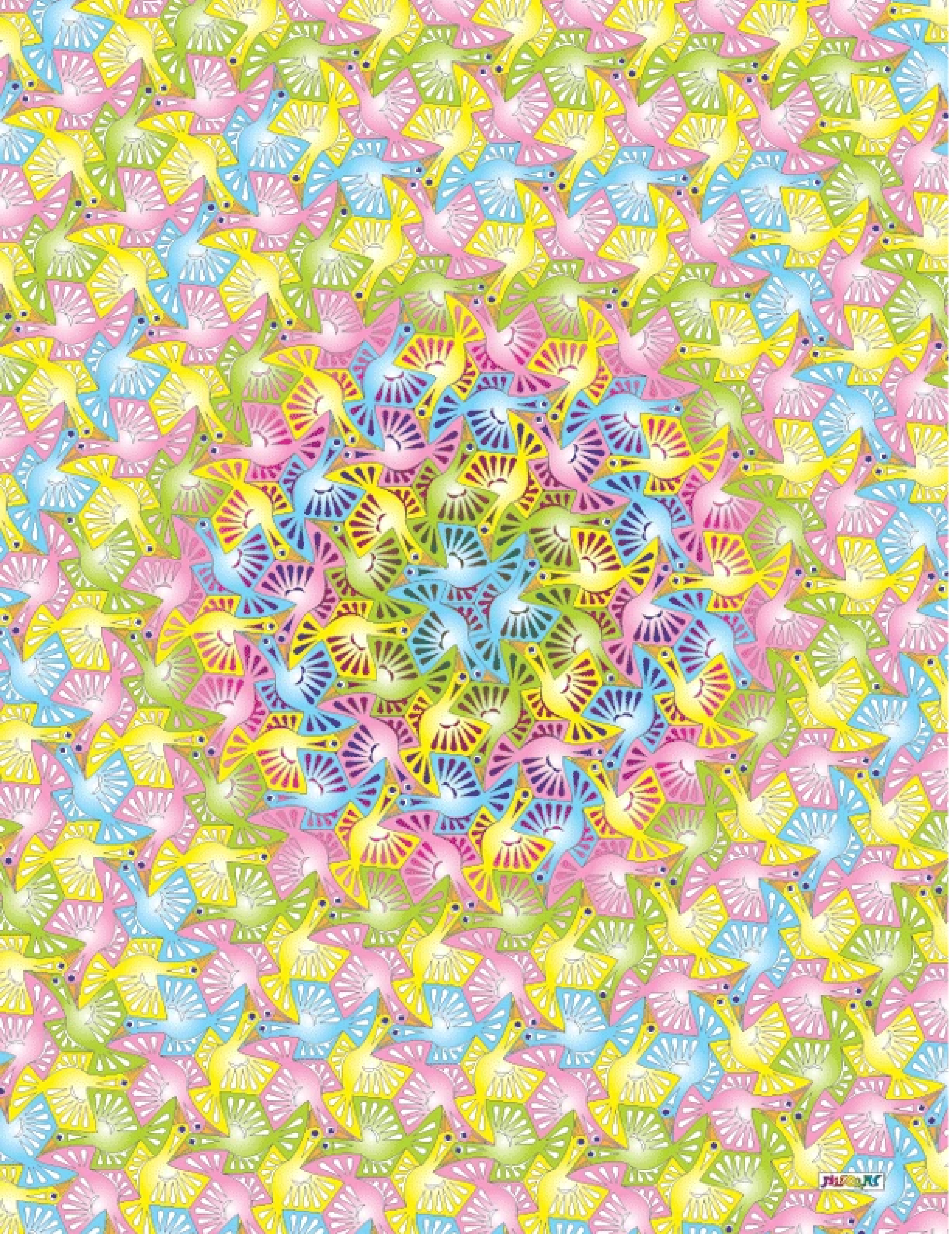


Statistics

Lecture 5

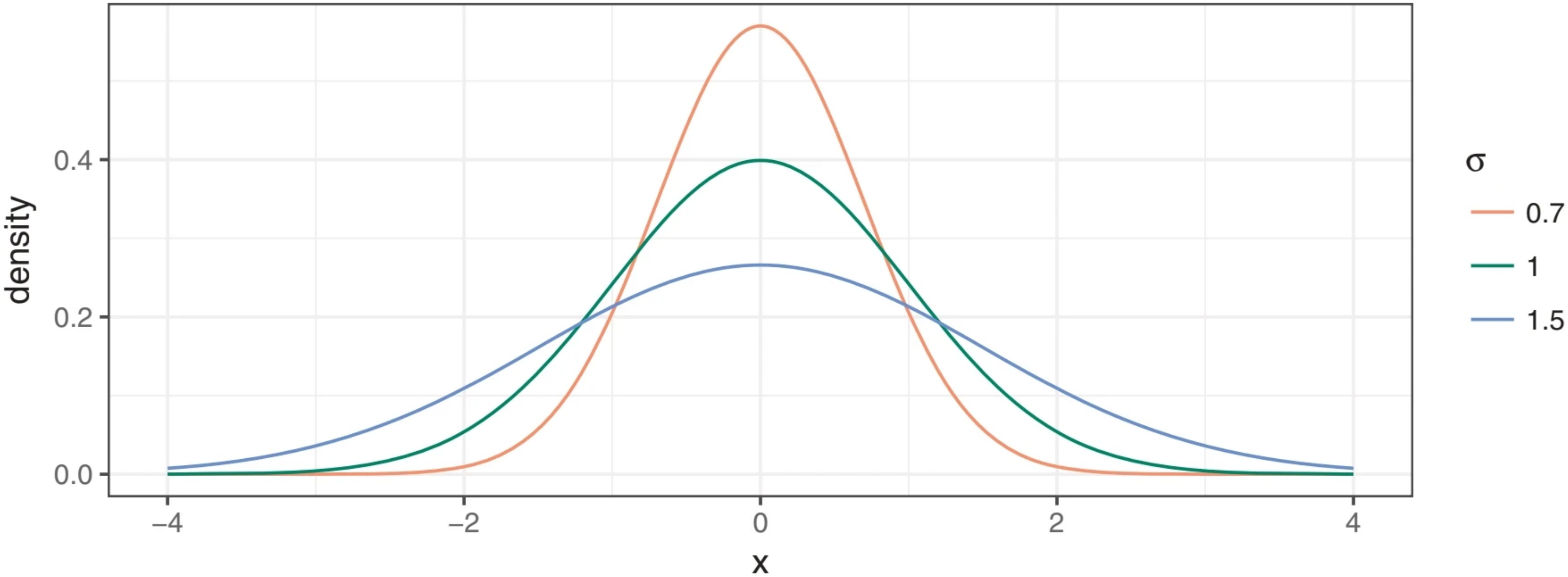
Distributions derived from normal distributions

Distributions derived from normal distributions

The density for normal distribution Normal(μ, σ^2) is given by $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Its m.g.f is

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}.$$


density
0.4
0.2
0.0
–4
–2
0
2
4
x
σ
0.7
1
1.5

χ^2 distribution

If Z is a standard normal distribution, the distribution $U = Z^2$ is called the χ^2 distribution with 1 degree of freedom.

Let us compute the p.d.f. for χ^2.

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$$F_U(x) = P(U \le x)$$

$$= P(-\sqrt{x} \le Z \le \sqrt{x})$$

$$= \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$$

where Φ is c.d.f. for Z.

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$$= P(-\sqrt{x} \leq Z \leq \sqrt{x})$$

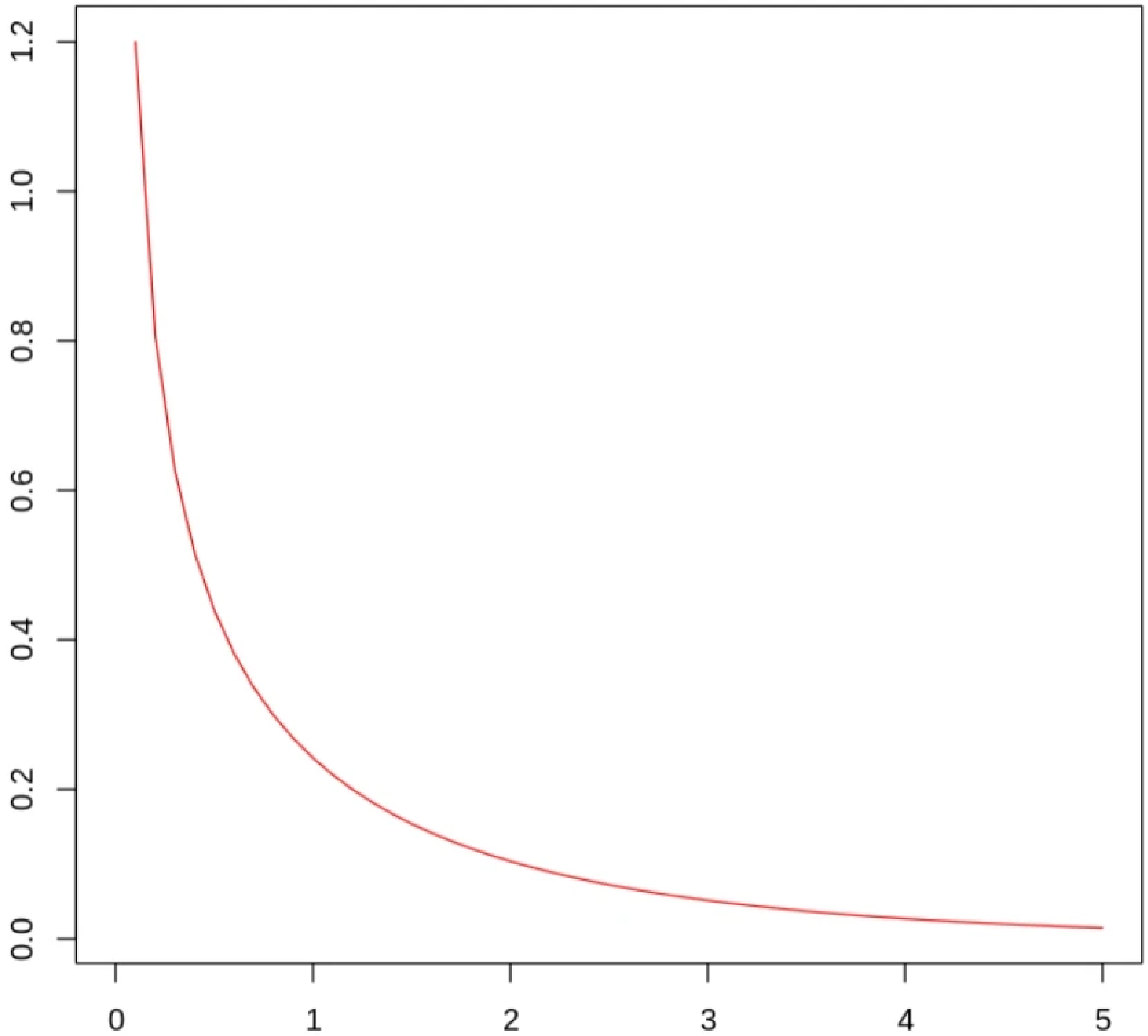
$$= \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$$

where Φ is c.d.f. for Z.

Differentiating w.r.t. x, we get

$$f_U(x) = \frac{1}{2} x^{-1/2} \phi(\sqrt{x}) + \frac{1}{2} x^{-1/2} \phi(-\sqrt{x})$$

$$= x^{-1/2} \phi(\sqrt{x}) = \frac{x^{-1/2} e^{-x/2}}{\sqrt{2\pi}}, \quad x \geq 0$$


1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5

If $U_1, U_2, \ldots, U_n$ are independent χ^2-distribution random variables with 1 degree of freedom, the random variable

$V = U_1 + U_2 + \cdots + U_n$ is called the χ^2 distribution with n degrees of freedom and written as χ^2_n

To compute the p.d.f. of χ_n^2, we need to know a little bit about the Gamma distribution and its p.d.f and m.g.f.

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$\boxed{\text{Gamma}(\alpha, \lambda)}$

The Gamma distribution depends on two parameters α and λ.

Its density is $\boxed{g(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}, \ t \geq 0}$

where $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du, \quad x > 0$

Notice that when $\alpha = 1$, the Gamma distribution coincides with the Exponential Distribution.

The parameter α is called a shape parameter and the parameter λ is called a scale parameter.

this is also called the rate parameter.

Changing λ changes the size of the distribution.

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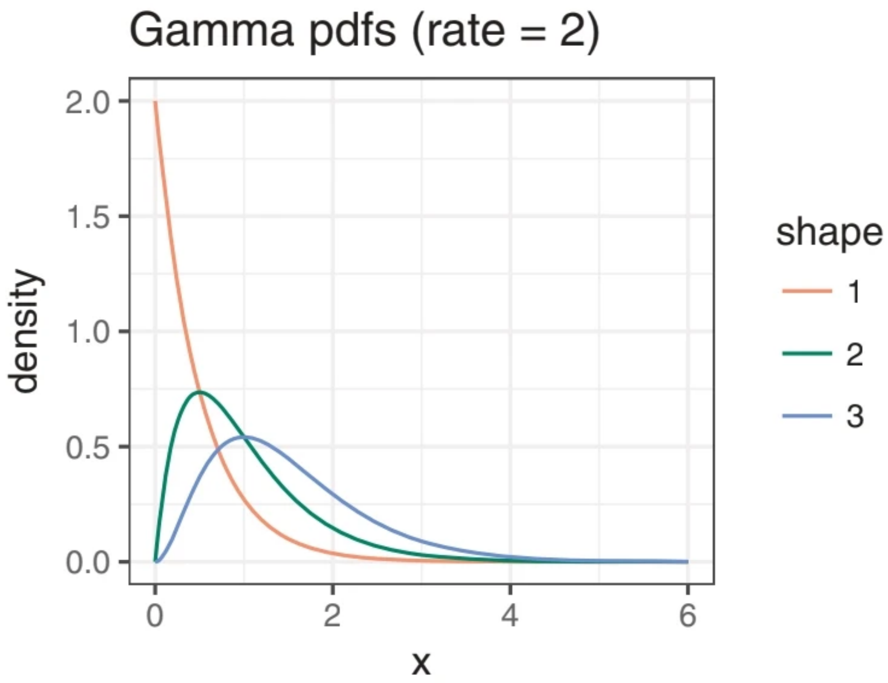
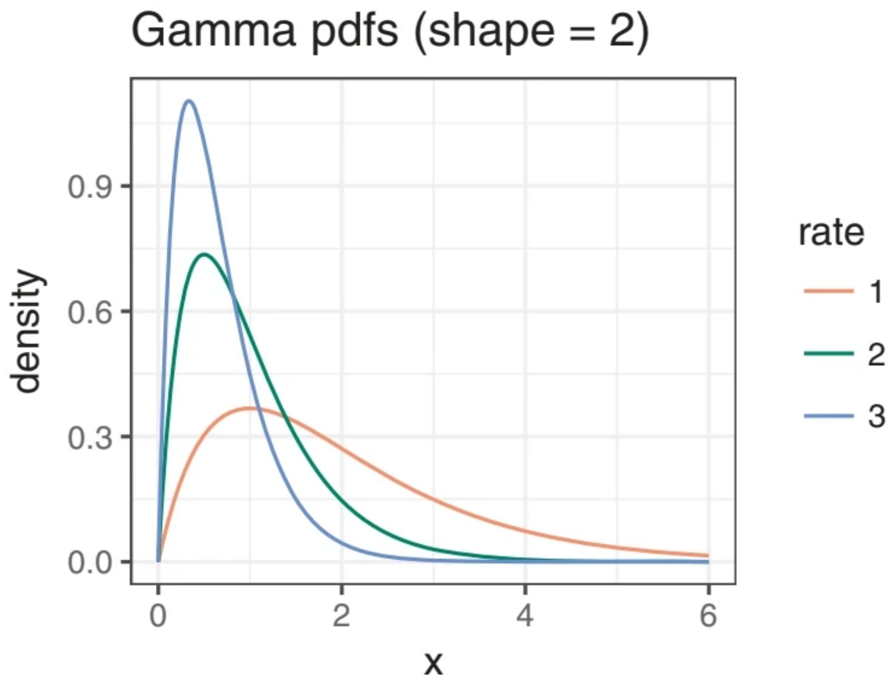
Changing λ changes the size of the distribution.

Changing α changes the shape of the distribution


Gamma pdfs (rate = 2)
density
2.0
1.5
1.0
0.5
0.0
0
2
4
6
x
shape
1
2
3


Gamma pdfs (shape = 2)
density
0.9
0.6
0.3
0.0
0
2
4
6
x
rate
1
2
3

Let us find the m.g.f of gamma dist.

$$M(t) = \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{(t-\lambda)x}\, dx$$

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Observe that Gamma density is

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda x}$$

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The integral evaluates to $\frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}$.

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$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{(t-\lambda)x}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$$ provided $\underline{\underline{\lambda > t}}$.

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$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha \text{ provided } \underline{\underline{\lambda > t}}.$$

Of course, this follows easily from the definition of Gamma function.

Since m.g.f s of sums are products of m.g.f s for independent r.v.s.

If $X_1 \sim Gamma(\alpha_1, \lambda)$ and $X_2 \sim Gamma(\alpha_2, \lambda)$ then $mgf(X_1+X_2)$

$$= mgf(X_1)\, mgf(X_2) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_1}\left(\frac{\lambda}{\lambda-t}\right)^{\alpha_2} = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_1+\alpha_2}$$

Since m.g.f s of sums are products of m.g.f^s for independent r.v.s. If $X_1 \sim \text{Gamma}(\alpha_1, \lambda)$ and $X_2 \sim \text{Gamma}(\alpha_2, \lambda)$ then $\text{mgf}(X_1 + X_2)$

$$= \text{mgf}(X_1)\,\text{mgf}(X_2) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1} \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2} = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1 + \alpha_2}$$

This is precisely the m.g.f. of $\text{Gamma}(\alpha_1 + \alpha_2, \lambda)$

Since m.g.f s of sums are products of m.g.f s for independent r.v.s.

therefore Sum of gammas are also gammas $\sim (\alpha_1+\alpha_2, \lambda)$

(α_1, λ) (α_2, λ)

Since m.g.f s of sums are products of m.g.f^{s} for independent r.v.s.

(α_1, λ) (α_2, λ)

therefore Sum of gammas are also gammas $\sim (\alpha_1+\alpha_2, \lambda)$

Notice that χ^2 which is given by the density

$$\frac{x^{-1/2} e^{-x/2}}{\sqrt{2\pi}}$$

is, in fact, a Gamma distribution $\text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

Therefore, χ_n^2 which is the sum of n χ^2 ($\text{Gamma}(1/2, 1/2)$) independent r.v.s has the $\boxed{\text{Gamma}(\alpha, \lambda)}$ distribution with $\alpha = n/2$, $\lambda = 1/2$

and therefore the density is

$$f(v) = \frac{1}{2^{n/2}\,\Gamma(n/2)} v^{n/2-1} e^{-v/2}, \quad v \geq 0.$$

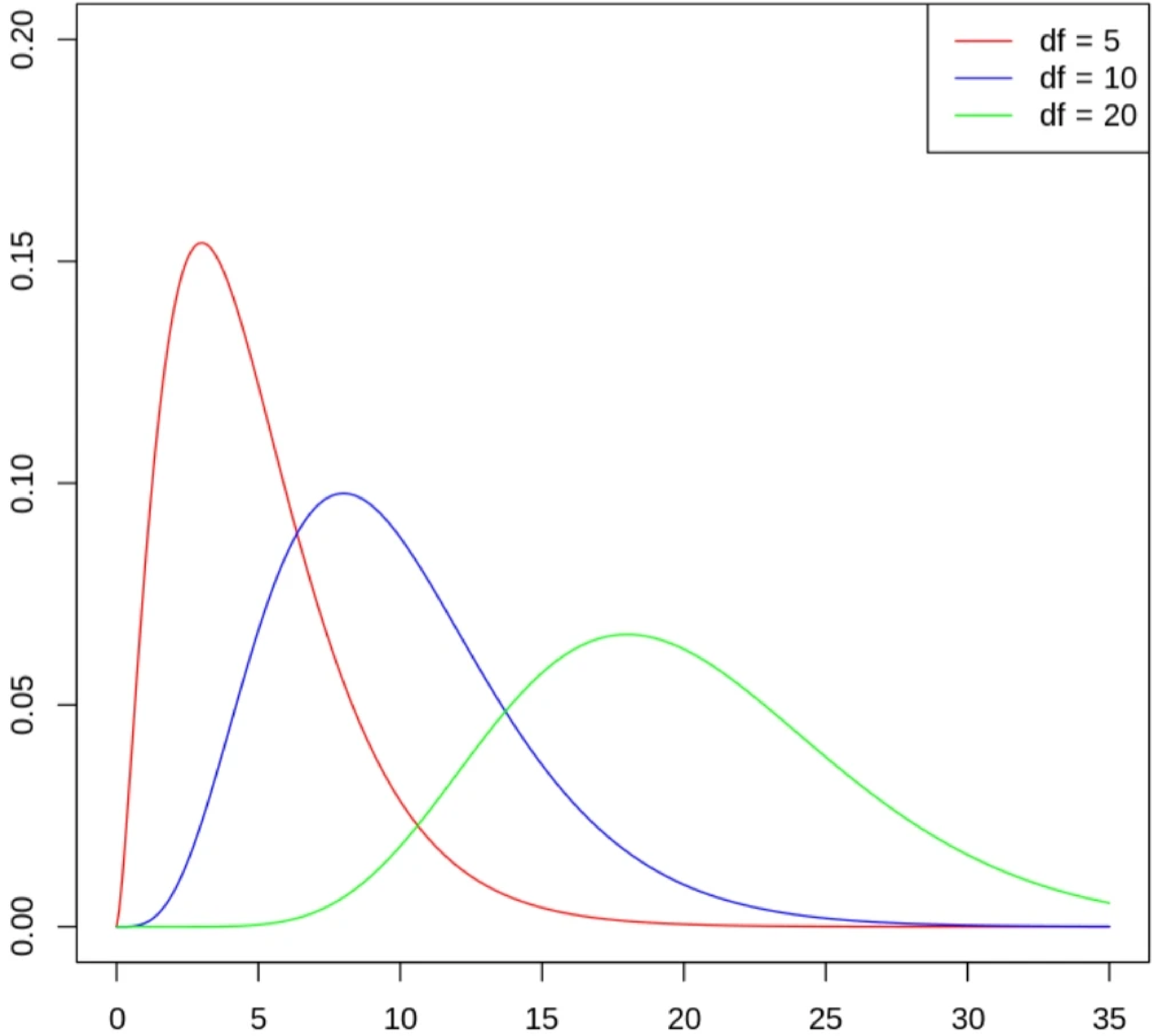
Therefore, χ_n^2 has gamma distribution with $\alpha = n/2$, $\lambda = 1/2$

and therefore the density is

$$f(v) = \frac{1}{2^{n/2}\,\Gamma(n/2)}\, v^{n/2 - 1} e^{-v/2}, \quad v \geq 0.$$

Its moment generating function is

$$M(t) = (1-2t)^{-n/2}$$


df = 5
df = 10
df = 20
0.00
0.05
0.10
0.15
0.20
0
5
10
15
20
25
30
35

Inverted Gamma

The density of Gamma (n, λ) is

$$f(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}, \quad 0 < x < \infty.$$

Inverted Gamma

The density of $X \sim \text{Gamma}(n, \lambda)$ is

$$f(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}, \quad 0 < x < \infty.$$

Suppose we want to know the density of the r.v. given by $Y = \frac{1}{X}$.

Inverted Gamma

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Suppose we want to know the density of the r.v. given by $Y = \frac{1}{X}$. Since X has support in $(0, \infty)$, so does Y.

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Suppose we want to know the density of the r.v. given by $Y = \frac{1}{X}$. Since X has support in $(0, \infty)$, so does Y.

$$F_Y(y) = P(Y \leq y) = P\left(1/X \leq y\right) = P\left(X \geq 1/y\right)$$

$$= \int_{1/y}^{\infty} f_X(x)\,dx = 1 - \int_0^{1/y} f_X(x)\,dx$$

Inverted gamma

The density of $X \sim \text{Gamma}(n, \lambda)$ is

$$f(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}, \quad 0 < x < \infty$$

The distribution of $Y = 1/X$ is given by:

$$F_Y(y) = P(Y \le y) = P(1/X \le y) = P(X \ge 1/y)$$

$$= \int_{1/y}^{\infty} f_X(x)\,dx = 1 - \int_0^{1/y} f_X(x)\,dx$$

We obtain the density by differentiating w.r.t. y:

$$f_Y(y) = \frac{1}{y^2} f_X(1/y) = \frac{\lambda^n}{(n-1)!} \left(\frac{1}{y}\right)^{n+1} e^{-\lambda/y}, \quad y > 0$$

F distribution

F distribution

Let $Z = U/m \,/\, V/n$ where

U and V are independent χ^2 random variables with m and n degrees of freedom respectively. Then Z is said to follow F distribution denoted $F_{m,n}$

We now derive its density.

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$$Z = \frac{U}{m} \Big/ \frac{V}{n} = \frac{U}{V} \cdot \frac{n}{m}$$

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<u>Step 1</u> Density of U/V

We now derive its density.

$$Z = \frac{U}{m} \Big/ \frac{V}{n} = \frac{U}{V} \cdot \frac{n}{m}$$

Step 1 Density of U/V

$P(U/V \leq z)$

this domain is split into two parts

$V < 0$ and $U \geq Vz$

$V > 0$ and $U \leq Vz$

We now derive its density.

$$Z = \frac{U}{m} \Big/ \frac{V}{n} = \frac{U}{V} \cdot \frac{n}{m}$$

Step 1 Density of U/V

$$P(U/V \leq z) = \int_{-\infty}^{0} \int_{vz}^{\infty} f(u,v)\,du\,dv + \int_{0}^{\infty} \int_{-\infty}^{vz} f(u,v)\,du\,dv$$

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$$P(U/V \le z) = \int_{-\infty}^{0} \int_{vz}^{\infty} f(u,v)\,du\,dv + \int_{0}^{\infty} \int_{-\infty}^{vz} f(u,v)\,du\,dv$$

(change of variables $vt = u$)

$$= \int_{-\infty}^{0} \int_{z}^{-\infty} v f(tv,v)\,dt\,dv + \int_{0}^{\infty} \int_{-\infty}^{z} v\, f(tv,v)\,dt\,dv$$

We now derive its density.

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Step 1 Density of U/V

$$P(U/V \le z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} |v| \, f(tv, v) \, dv \, dt$$

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Step 1 Density of U/V

$$P(U/V \le z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} |v| f(tv, v)\, dv\, dt$$

Differentiating w.r.t. z, we get

$$f_{U/V}(z) = \int_{-\infty}^{\infty} |v| f(zv, v)\, dv$$

We now derive its density.

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Step 1 Density of U/V

$$P(U/V \le z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} |v| f(tv, v)\, dv\, dt$$

Differentiating w.r.t. z, we get

$$f_{U/V}(z) = \int_{-\infty}^{\infty} |v| f_U(zv) f_V(v)\, dv.$$

Therefore, the density of u/v is

$$f(z) = \int_0^\infty v \frac{1}{2^{m/2}\Gamma(m/2)} (zv)^{m/2-1} e^{-\frac{zv}{2}} \cdot \frac{1}{2^{n/2}\Gamma(n/2)} v^{n/2-1} e^{-v/2} dv$$

$$= \frac{z^{m/2-1}}{2^{\frac{n+m}{2}}\Gamma(\frac{m}{2})\Gamma(n/2)} \int_0^\infty v^{\frac{m}{2}+\frac{n}{2}-1} e^{-\frac{1}{2}(1+z)v} dv$$

$$= \frac{z^{m/2-1}}{(1+z)^{m/2+n/2}\, 2^{\frac{n+m}{2}}\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \int_0^\infty t^{\frac{m}{2}+\frac{n}{2}-1} e^{-t/2} dt$$

Therefore, the density of U/V is

$$f(z) = \int_0^\infty v \frac{1}{2^{m/2}\Gamma(m/2)} (zv)^{m/2-1} e^{-\frac{zv}{2}} \cdot \frac{1}{2^{n/2}\Gamma(n/2)} v^{n/2-1} e^{-v/2} dv$$

$$= \frac{z^{m/2-1}}{2^{\frac{n+m}{2}} \Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_0^\infty v^{\frac{m}{2}+\frac{n}{2}-1} e^{-\frac{1}{2}(1+z)v} dv$$

$$\boxed{= \frac{z^{m/2-1}}{(1+z)^{m/2+n/2}} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}}$$

Step 2 Density of $Z = \frac{U}{V} \cdot \frac{n}{m}$

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$$P(Z \leq z) = P\left(\frac{U}{V} \cdot \frac{n}{m} \leq z\right)$$

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$$P(Z \le z) = P\left(\frac{U}{V} \cdot \frac{n}{m} \le z\right)$$

$$= P\left(\frac{U}{V} \le \frac{mz}{n}\right)$$

$$= \int_{-\infty}^{\frac{mz}{n}} f_{\frac{U}{V}}(w)\, dw$$

$$\therefore f_Z(z) = \frac{m}{n} f_{\frac{U}{V}}\left(\frac{mz}{n}\right)$$

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$$\therefore f_Z(z) = \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} z^{\frac{m}{2}-1}}{\left(1+\frac{m}{n}z\right)^{\frac{m+n}{2}}} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

Step 2 Density of $Z = \frac{U}{V} \cdot \frac{n}{m}$

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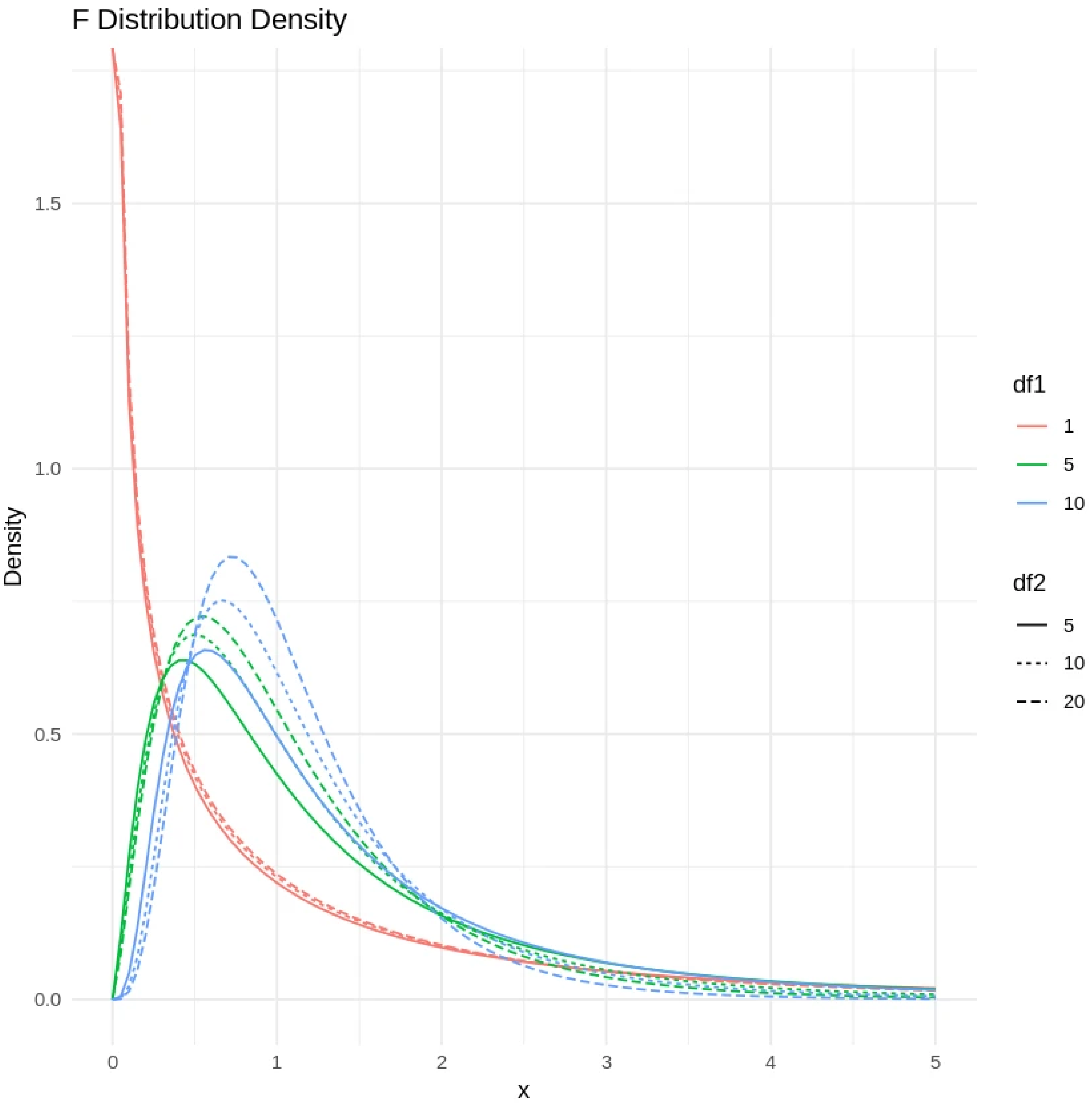
$F_{m,n}$ **distribution**

$$\therefore f_Z(z) = \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} z^{\frac{m}{2}-1}}{\left(1+\frac{m}{n}z\right)^{\frac{m+n}{2}}} \quad \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$


F Distribution Density
Density
1.5
1.0
0.5
0.0
0
1
2
3
4
5
x
df1
1
5
10
df2
5
10
20

t distributions

Let Z be a Normal $(0,1)$ r.v. and let U be a χ_n^2 r.v. Then $Y = \dfrac{Z}{\sqrt{U/n}}$

is called the t - distribution with n degrees of freedom.

We compute the density of the <u>t distribution</u>

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Let $H = Y^2 = \frac{Z^2}{U/n}$ where Z^2 has χ^2_1 distribution. Therefore, H has $F_{1,n}$ distribution!

We compute the density of the $\underline{t \text{ distribution}}$

Let $H = Y^2 = \dfrac{Z^2}{U/n}$ where Z^2 has χ_1^2 distribution. Therefore, H has $F_{1,n}$ distribution!

The density of H is given by

$$f_H(u) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma(n/2)} \frac{u^{-1/2}}{\left(1+\frac{u}{n}\right)^{\frac{n+1}{2}}}.$$

Now notice that

$$P(H \leq u) = P(Y^2 \leq u)$$
$$= P(-\sqrt{u} \leq Y \leq \sqrt{u})$$
$$= P(Y \leq \sqrt{u}) - P(Y \leq -\sqrt{u})$$
$$= P(Y \leq \sqrt{u}) - P(Y \geq \sqrt{u})$$

(since Y is even)

$$= 2P(Y \leq \sqrt{u}) - 1$$

$$\therefore f_H(u) = \frac{1}{\sqrt{u}} f_Y(\sqrt{u})$$

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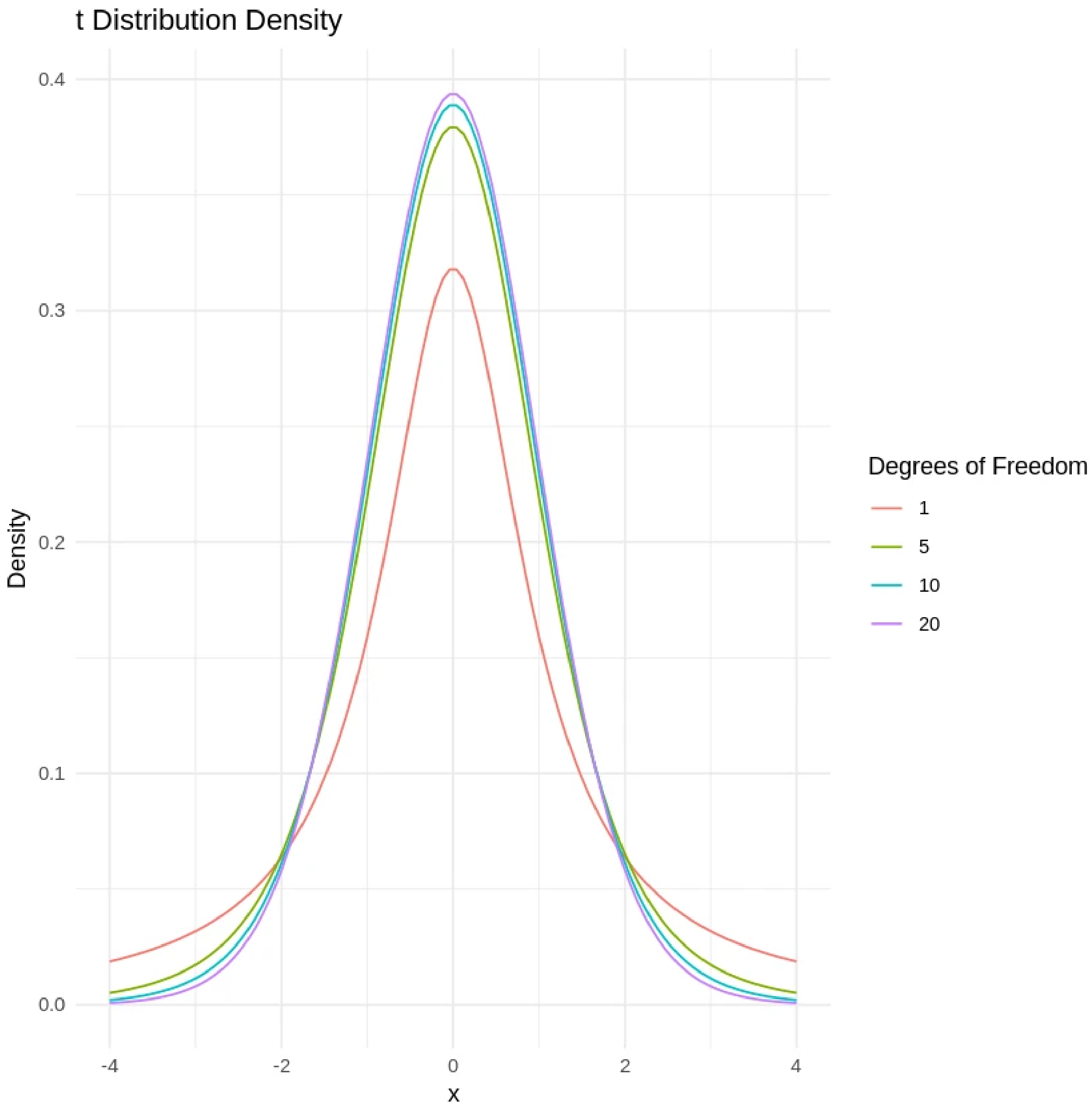
$$= P(Y \le \sqrt{u}) - P(Y \ge \sqrt{u})$$

$$= 2\,P(Y \le \sqrt{u}) - 1$$

$$\therefore\ f_H(u) = \frac{1}{\sqrt{u}}\, f_Y(\sqrt{u})$$

$$\therefore\ f_Y(z) = |z|\, f_H(z^2) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1 + \frac{z^2}{n}\right)^{-\frac{n+1}{2}}$$


t Distribution Density
0.4
0.3
0.2
0.1
0.0
Density
-4
-2
0
2
4
x
Degrees of Freedom
1
5
10
20