STATISTICS

MATH 414

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Given an uncountable sample space Ω, singular outcomes $\{x\}$ for $x \in \Omega$ cannot be assigned positive probabilities.

Moreover, any "reasonable" notion of probability will not allow that <u>every</u> subset of Ω be considered admissible as an <u>event</u>.

Therefore, we must restrict ourselves to collection of events called σ-algebras $\mathcal{F}$ s.t.

(i) The sample space $\Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, so does A^c

(iii) If $\{E_j\}_{j \in \mathbb{N}}$ is a disjoint collection of events then their union $\bigcup_{j \in \mathbb{N}} E_j$ is also an event.

It is possible and useful to define a probability on $\mathcal{F}$.

$P : \mathcal{F} \longrightarrow [0,1]$ is a **probability** if

(i) $P(\Omega) = 1$

(ii) If $(E_j)_{j \in \mathbb{N}}$ is a disjoint collection of events then

$$P\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} P(E_j)$$

The triplet

$$\boxed{(\Omega, \mathcal{F}, P)}$$ is called a

Probability space.

Example

① $\Omega = \mathbb{R}$

The intervals (a, b) or $[a, b)$ do not form a σ-algebra but they can "generate" one. It is therefore enough to define a probability on them

$$P((a,b)) = \text{length of } ((a,b) \cap (0,1))$$

The probability from the previous example can be written as

$$P((a,b)) = \int_{(a,b)\cap(0,1)} 1 \, dx$$

$$= \int_{(a,b)} f(x)\, dx$$

where

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1) \end{cases}$$

This function f is called a density function and it satisfies the properties

(i) $f : \mathbb{R} \to [0,1]$, $f(x) \geq 0 \ \forall x$

(ii) f is piecewise-continuous

(iii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

A density function defines a probability P on $\mathbb{R}$ by $P(E) = \int_E f(x)\,dx$ for $E \in \mathcal{F}$ some σ-algebra on $\mathbb{R}$

Continuous Random Variables.

Continuous Random Variables.

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Let $X : \Omega \to \mathbb{R}$ be a function. X is a random variable if B is an event in $\mathbb{R}$ $\implies$ $X^{-1}(B)$ is an event in Ω.

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Let $X : \Omega \to \mathbb{R}$ be a function. X is a random variable if B is an event in $\mathbb{R}$ $\Rightarrow$ $X^{-1}(B)$ is an event in Ω.

B should be a Borel set – enough to check with intervals of the form (a, ∞) for $a \in \mathbb{R}$.

Probability Density function.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
A random Variable $X : \Omega \to \mathbb{R}$ is called a continuous random variable if there exists a density function $f_X : \mathbb{R} \to \mathbb{R}$ s.t. for $A \in \mathcal{F}$,

$$\boxed{\mathbb{P}(X \in A) = \int_A f_X(x)\, dx}$$

f_X is called the probability density function of X.

The function

$$F(x) := P(X \leq x) = \int_{-\infty}^{x} f_X(\sigma)\, d\sigma$$

is called the cumulative distribution function of X.

Well known Distributions.

$X \sim \text{Uniform}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in (a, b) \\ 0 & \text{if } x \notin (a, b) \end{cases}$$

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$$F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x > b \end{cases}$$

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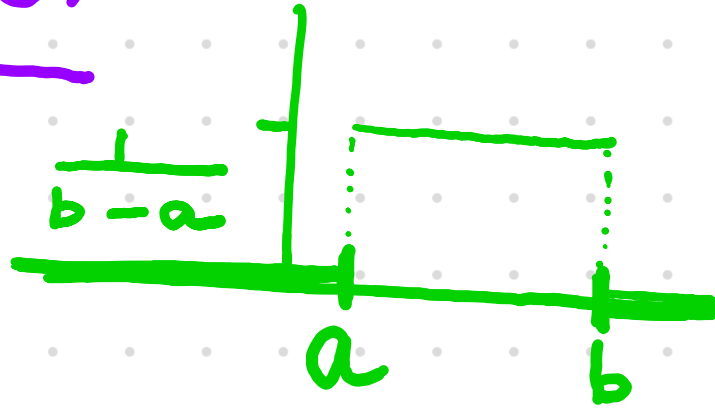

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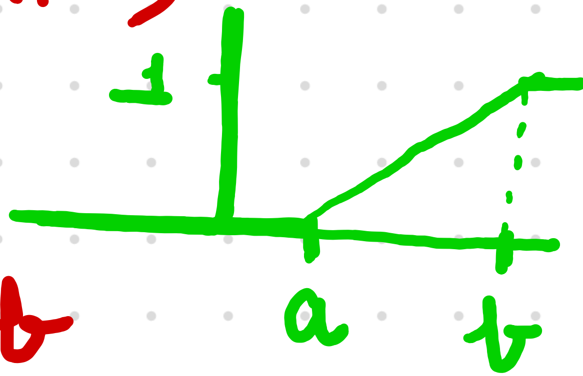

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$X \sim Exp(\lambda)$

Let $\lambda > 0$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

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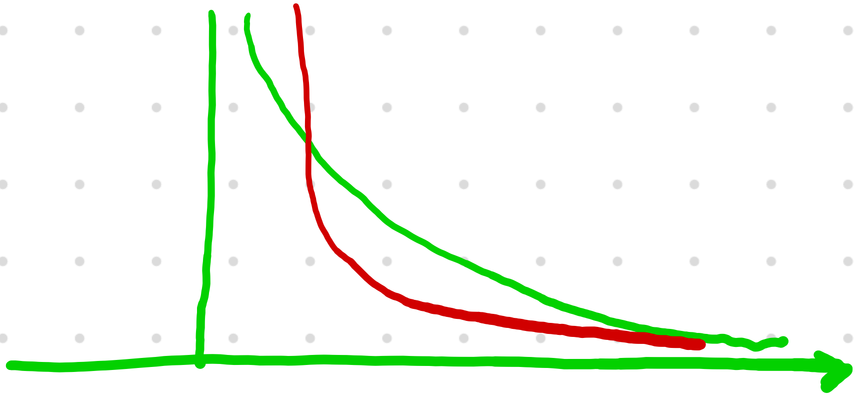

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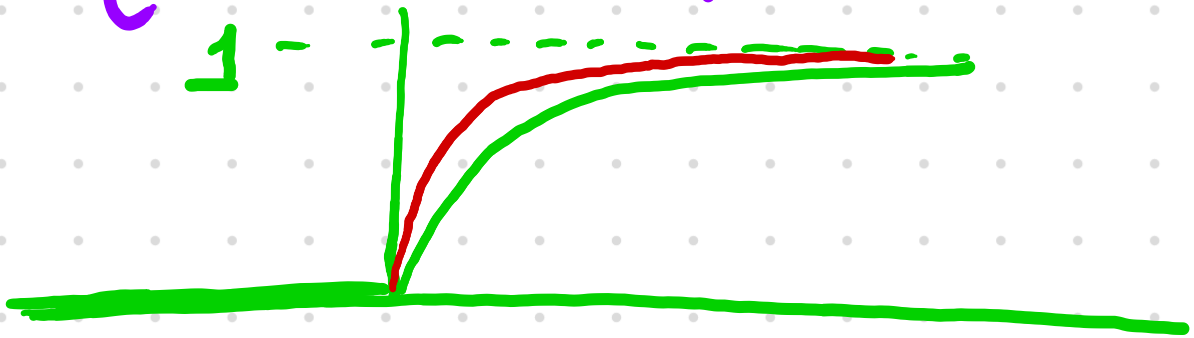
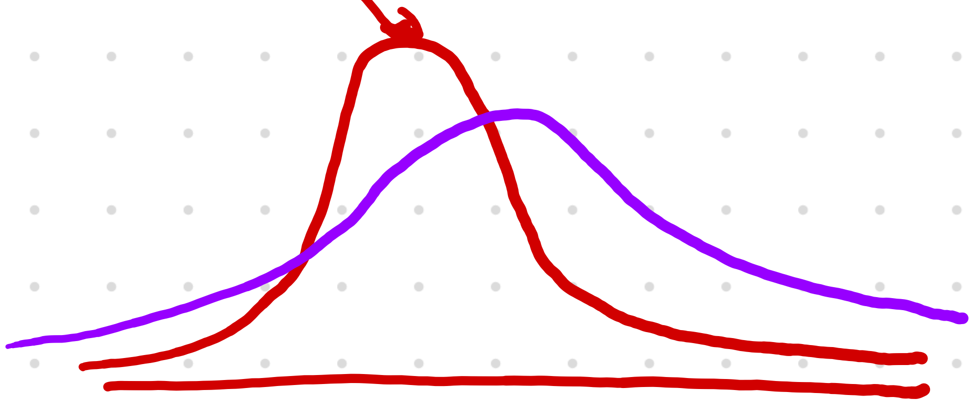
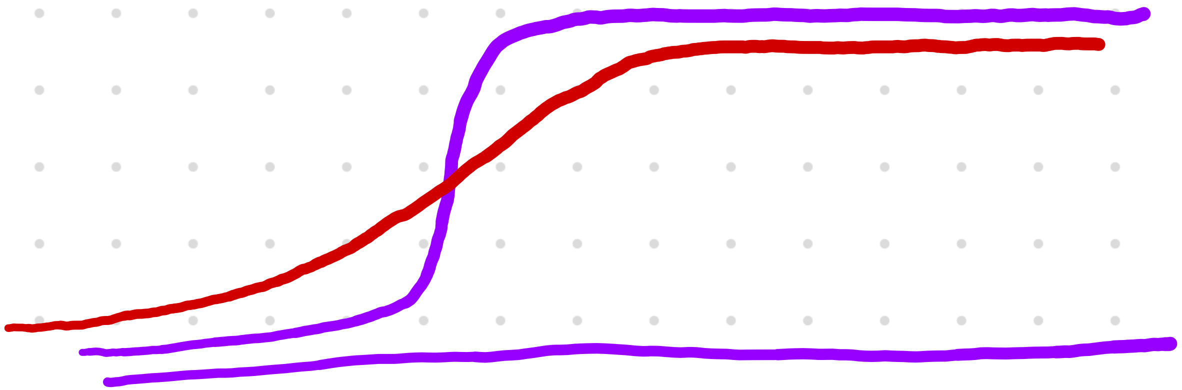


$X \sim \text{Normal}(\mu, \sigma^2), \quad \mu \in \mathbb{R}, \quad \sigma > 0.$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$



$$F(x) = \int_{-\infty}^{x} f_X(z)\,dz$$



Multiple Continuous Random Variables

Multiple Continuous Random Variables

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a non-negative function, piecewise continuous in each variable and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$.

For a Borel set $A \subset \mathbb{R}^2$, define

$$\mathbb{P}(A) = \iint_A f(x,y)\, dx\, dy.$$

Then $\mathbb{P}$ is a probability on $\mathbb{R}^2$ with density f.

A pair of random variables (X,Y) is said to have a joint density $f(x,y)$ if for every Borel set $A \subset \mathbb{R}^2$

$$\boxed{P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy}$$

The function

$$F(x,y) = P((X \leq a) \cap (Y \leq b))$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{b} f(z,w)\, dz\, dw$$

is called the joint distribution function of X, Y.

Ex. 1

$$f(x,y) = \begin{cases} 1/2 & \text{if } (x,y) \in (0,1) \times (3,5) \\ 0 & \text{otherwise.} \end{cases}$$

Then if $A = (a,b) \times (c,d) \subseteq (0,1) \times (3,5)$

then $P(A) = \displaystyle\int_{-\infty}^{d}\int_{-\infty}^{b} f(x,y)\,dx\,dy$

$$= \frac{(b-a)(d-c)}{2} = \frac{\text{area of } A}{\text{area of } (0,1)\times(3,5)}$$

ex.2

$$f(x,y) = \begin{cases} x+y & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

$$P(\mathbb{R}^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x+y \, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^1 dy$$

$$= \int_0^1 \frac{1}{2} + y \, dy = \left[y^2/2 + y/2 \right]_0^1$$

$$= 1.$$

Exercise

In the last example,

Calculate

$$P\left(X < \frac{1}{2}, Y < \frac{1}{2}\right)$$

$$P(X < 1/2)$$

$$P(Y < 1/2)$$

and observe that the random variables X and Y are not independent.

Marginal Distributions

If the random variables X and Y have the joint density $f(x,y)$, we can compute densities for the random variables X and Y

$$P(X \le x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,y)\, dy\, du$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$ is the density for X.

Similarly $h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$ is the density for Y.

These distributions are called the marginal distributions.

Exercise: Check that the marginal distributions for the uniform distribution of $(0,1) \times (3,5)$ are both uniform and hence the two random variables are independent.

Ex.

Consider the disk in $\mathbb{R}^2$ given by

$$C = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 25 \}$$

$\therefore$ Area of $C = 25\pi$.

Define the joint density

$$f(x,y) = \begin{cases} 1/25\pi & \text{if } x \in C \\ 0 & \text{otherwise.} \end{cases}$$

It may be checked easily that given $A \subseteq \mathbb{R}^2$ (Borel)

$$P(A) = \text{Area of } A \Big/ \text{Area of } C.$$

Let us look at the marginal distribution

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \begin{cases} \frac{2}{25\pi}\sqrt{25-x^2} & \text{if } x \in (-5,5) \\ 0 & \text{otherwise.} \end{cases}$$

This is called the <u>semi-circular law.</u>

Conditional Density

Let (X, Y) be random variables with joint density $f(x,y)$. Let the marginal density of Y be $f_Y(\cdot)$. Suppose b is a real number such that $f_Y(b) > 0$. The conditional density of X given $Y=b$ is given by

$$f_{X|Y=b} = \frac{f(x,b)}{f_Y(b)}, \quad x \in \mathbb{R}.$$

Similarly if the marginal density of X is $f_X(\cdot)$ s.t. $f_X(a) > 0$ for some $a \in \mathbb{R}$. Then the conditional density of Y given $X = a$ is given by

$$f_{Y|X=a}(y) = \frac{f(a,y)}{f_X(a)}, \quad y \in \mathbb{R}.$$

If X and Y are independent then

$$f_{X|Y=b}(x) = \frac{f(x,b)}{f_Y(b)} = \frac{f_X(x)\, f_Y(b)}{f_Y(b)}$$

$$= f_X(x).$$

We can use conditional densities to compute conditional probabilities.

$$P(X \in A \mid Y = b) = \int_A \frac{f(x,b)}{f_Y(b)}\,dx$$

Example 1 Let (X, Y) have joint p.d.f

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{1}{2}(x^2 - xy + y^2)}, \quad (x,y) \in \mathbb{R}^2$$

Marginal density of X

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the marginal distribution for Y looks similar.

Compare $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Therefore, X and Y are not independent.

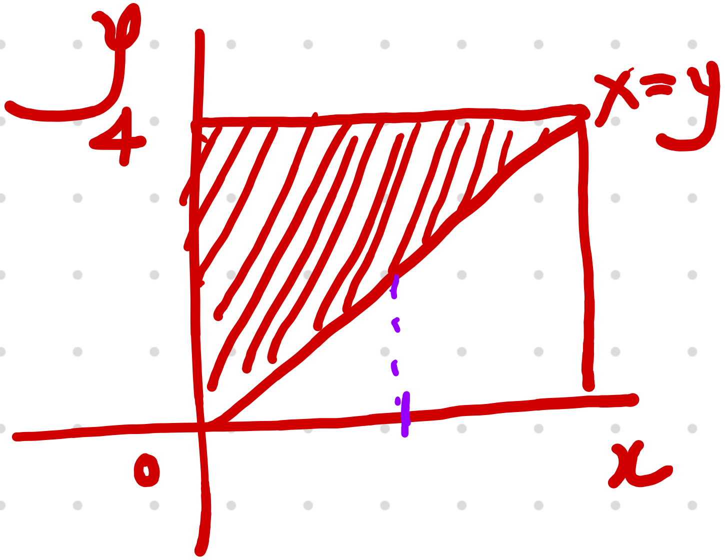
As a result, their conditional distributions will be different from their marginals.

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{1}{\sqrt{2\pi}} e^{-1/2\,(y - x/2)^2}, \quad y \in \mathbb{R}.$$

Exercise 2

Consider the uniform distribution on

$$T = \{(x,y) : 0 < x < y < 4\}$$



$$f(x,y) = \begin{cases} 1/8 & \text{for } (x,y) \in T \\ 0 & \text{otherwise.} \end{cases}$$

Find marginal and conditional densities.

Expectation

Expectation

Let X be a continuous random variable with piecewise continuous density $f(x)$.

The expectation of X is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx.$$

when the integral converges absolutely.

In this case, we say that X has "finite expectation".

Example 1

$$X \sim \text{Uniform}(a, b)$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}.$$

The expectation is the midpoint of the interval.

Example 2 Let $0 < \alpha < 1$, $X \sim \text{Pareto}(\alpha)$

then $f(x) = \begin{cases} \dfrac{\alpha}{x^{\alpha+1}} & 1 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$

then $E[X] = \displaystyle\int_1^\infty \frac{x \cdot \alpha}{x^{\alpha+1}}\, dx = \alpha \lim_{M \to \infty} \int_1^M x^{-\alpha}\, dx$

$= \infty.$

Thus the Pareto r.v. has infinite expectation.

Example 3

Let $X \sim$ Cauchy$(0,1)$. Then its p.d.f. is given by

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$\therefore \quad E[X] = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x}{1+x^2}\, dx$$

In this case, the $\lim\limits_{\substack{M \to -\infty \\ N \to \infty}} \int_{M}^{N} \frac{x}{1+x^2}\, dx$

is not defined.

Let X be a continuous r.v. with p.d.f. $f_X : \mathbb{R} \to \mathbb{R}$.

Let g be a piecewise cts f^n, $g : \mathbb{R} \to \mathbb{R}$ then $Z = g(X)$ has expectation

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

where the c.d.f. of Z is defined as

$$F(z) = P(Z \le z) = P(g(X) \le z).$$

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$$F(z) = P(Z \le z) = P(g(X) \le z).$$

(This is a change of variables formula from single variable calculus and a proof will not be found in this course).

Similarly let (X,Y) have joint p.d.f. $f: \mathbb{R}^2 \to \mathbb{R}$. and suppose that $h: \mathbb{R}^2 \to \mathbb{R}$ is Piecewise Continuous, then

$$E[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dx\, dy.$$

Example. Let $X, Y \sim \text{Uniform}(0,1)$.

What is the expected value of

$Z = \max\{X, Y\}$?

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method 1

The c.d.f. of Z is

$$F(z) = P(Z \le z)$$
$$= P(\max\{X,Y\} \le z)$$
$$= P(X \le z \text{ and } Y \le z)$$
$$(\because \text{ind.}) \quad = P(X \le z)\, P(Y \le z) = z^2$$

Thus its p.d.f is $f(z) = 2z$.

$$\therefore \quad E[Z] = \int_{-\infty}^{\infty} z\, f(z)\, dz$$

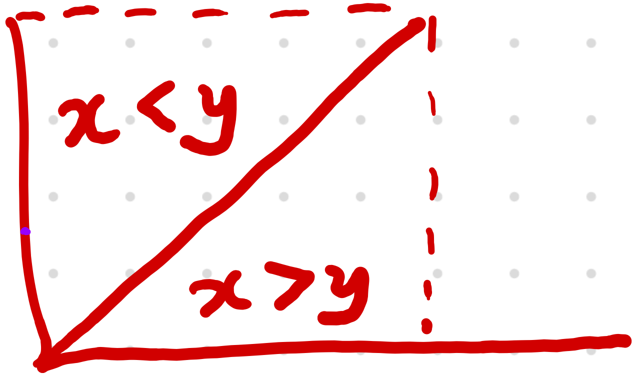
$$= \int_0^1 2z^2\, dz = \left[\frac{2z^3}{3}\right]_0^1$$

$$= 2/3.$$

method 2

$$E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max\{x,y\}\, f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 \max\{x,y\}\, dx\, dy$$



$$= \int_0^1\int_0^y y\, dx\, dy + \int_0^1\int_y^1 x\, dx\, dy$$

$$= 2/3.$$

Theorem. Suppose that X, Y are cts. r.v.s with joint density function $f: \mathbb{R}^2 \to \mathbb{R}$. Assume that both have finite expectation. let $a, b \in \mathbb{R}$:

(i) $E[aX] = aE[X]$

(ii) $E[aX+b] = aE[X]+b$

(iii) $E[X+Y] = E[X] + E[Y]$

(iv) $E[aX+bY] = aE[X] + bE[Y]$

(v) If $X \geq 0$ then $E[X] \geq 0$.

Let X be a random variable with p.d.f. $f: \mathbb{R} \to \mathbb{R}$. Suppose X has finite expectation.

The Variance of X is

$$Var[X] = E[(X - E[X])^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$SD[X] = \sqrt{Var[X]}$$

Let $a \in \mathbb{R}$, X a cts. r.v. with finite variance.

Then

(i) $Var[X] = E[X^2] - E[X]^2$

(ii) $Var[aX] = a^2 \cdot Var[X]$

(iii) $SD[aX] = |a| \, SD[X]$

(iv) $Var[X+a] = Var[X]$

(v) $SD[X+a] = SD[X]$

If X and Y are independent

(a) $E[XY] = E[X]E[Y]$

(b) $Var[X+Y] = Var[X] + Var[Y]$.

Example.

Let $X \sim$ Normal $(0,1)$

then its density is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

We want to compute its expectation and variance.

First, we claim that X has a finite expectation, i.e.,

$$\int_{-\infty}^{\infty} |x| e^{-x^2/2} \, dx < \infty$$

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This is easy to see since

$$\int_{-\infty}^{0} |x| e^{-x^2/2} dx = \lim_{M \to -\infty} \int_{M}^{0} -x e^{-x^2/2} dx \qquad \{-x = y\}$$

$$= \lim_{M \to \infty} \int_{0}^{M} y e^{-y^2/2} dy = \lim_{M \to \infty} \left[-e^{-y^2/2} \right]_0^M$$

$$= 1.$$

Similarly for $\int_0^\infty x e^{-x^2/2}\,dx$.

Therefore, X has expectation $E[X] = 0$.

Similarly, X has finite variance

$$\int_{-\infty}^{\infty} \frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}}\,dx = \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-x^2/2}\,dx$$

$$= 1$$ (Check using integration by parts)

Exercise.

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Let $Y \sim \text{Normal}(\mu, \sigma^2)$

Define $W = \dfrac{Y-\mu}{\sigma}$.

Check that $W \sim \text{Normal}(0,1)$.

Hence show that

$E[Y] = \mu$ and $\text{var}[X] = \sigma^2$.

Covariance

Covariance

Let X and Y be cts r.v.s with joint p.d.f. $f: \mathbb{R}^2 \to \mathbb{R}$. Suppose X and Y have finite expectation. The Covariance of X and Y is defined as

$$\mathrm{Cov}[X,Y] = E((X-\mu_X)(Y-\mu_Y))$$ where

$\mu_X = E[X]$

$\mu_Y = E[Y]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy.$$

Theorem Let X and Y be cts. r.v.s such that they have a joint p.d.f. Assume

$0 \neq \sigma_x^2 = Var[X] < \infty$, $0 \neq \sigma_y^2 = Var[Y] < \infty$.

Then

(i) $Cov[X, Y] = E[XY] - E[X]E[Y]$

(ii) $Cov[X, Y] = Cov[Y, X]$

(iii) $Cov[X, X] = Var[X]$

(iv) $-\sigma_x \sigma_y \leq Cov[X, Y] \leq \sigma_x \sigma_y$

(v) If X and Y are independent then

$Cov[X, Y] = 0$.

(vi) Covariance is linear in first & second variables.

Theorem Let X and Y be cts. r.v.s such that they have a joint p.d.f. Assume
$0 \neq \sigma_x^2 = \mathrm{Var}[X] < \infty$, $0 \neq \sigma_y^2 = \mathrm{Var}[Y] < \infty$.

Then

(i) $\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y]$

(ii) $\mathrm{Cov}[X,Y] = \mathrm{Cov}[Y,X]$

(iii) $\mathrm{Cov}[X,X] = \mathrm{Var}[X]$

look at
$$E\left[\left(\frac{X-\mu_X}{\sigma_X} \pm \frac{Y-\mu_Y}{\sigma_Y}\right)^2\right]$$

(iv) $-\sigma_x\sigma_y \leq \mathrm{Cov}[X,Y] \leq \sigma_x\sigma_y$

(v) If X and Y are independent then
$\mathrm{Cov}[X,Y] = 0$.

(vi) Covariance is linear in first & second variables.

Let X and Y be continuous random variable with a joint p.d.f., and both with finite variance and covariance.

The quantity $\rho[X,Y] = \dfrac{\operatorname{cov}[X,Y]}{\sigma_X \sigma_Y}$ is known as the correlation of X and Y.

Example.

$X \sim \text{Uniform}(0,1)$, $Y \sim \text{uniform}(0,1)$

X and Y are independent.

let $U = \min(X, Y)$ and $V = \max(X, Y)$.

let us find $f(U, V)$.

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let us find $f(U, V)$.

① $P(U \le u) = 1 - P(U > u)$

$= 1 - P(X > u \text{ and } Y > u)$

$= 1 - (1-u)^2$.

$\therefore f_U(u) = 2(1-u)$ on $(0,1)$

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let $U = \min(X,Y)$ and $V = \max(X,Y)$.

let us find $f(u,v)$.

2 $P(V \le v) = v^2$ (calculated earlier)

$\therefore f_V(v) = 2v$ on $(0,1)$

The joint distribution for U and V

$F(u,v) = P(U \leq u, V \leq v)$

$= P(V \leq v) - P(V \leq v, U > u)$

$= v^2 - P(u < X \leq v, u < Y \leq v)$

$= v^2 - P(u < X \leq v) P(u < Y \leq v)$

$= v^2 - (v-u)^2$

when $0 < u < v < 1$

The p.d.f. for the joint distribution is

$$f(u,v) = \frac{\partial^2 F}{\partial u \partial v} = \begin{cases} 2 & \text{if } 0<u<v<1 \\ 0 & \text{otherwise.} \end{cases}$$

With this, we can now calculate the correlation of U and V.

Conditional Expectation and Variance.

Let (X,Y) be continuous random variable with a piecewise continuous joint p.d.f. $f: \mathbb{R}^2 \to \mathbb{R}$. Let f_X be the marginal density of X.
Assume that $x \in \mathbb{R}$ satisfies $f_X(x) \neq 0$.
The conditional expectation of Y given $X = x$ is given as

$$E[Y \mid X=x] = \int_{-\infty}^{\infty} y \, f_{Y|X=x}(y)\, dy$$

$$= \int_{-\infty}^{\infty} y \, \frac{f(x,y)}{f_X(x)} \, dy.$$

The conditional variance of Y given $X=x$ is defined by

$$\mathrm{Var}[Y|X=x] = E\left[(Y-E[Y|X=x])^2 \mid X=x\right]$$

$$= \int_{-\infty}^{\infty} \left(y - \int_{-\infty}^{\infty} y \frac{f(x,y)}{f_X(x)}\, dy \right)^2 \frac{f(x,y)}{f_X(x)}\, dy .$$

Theorem Let (X, Y) be cts r.v. with joint p.d.f. $f : \mathbb{R}^2 \to \mathbb{R}$. Let

$$g(y) = \begin{cases} E[X \mid Y=y] & \text{if } f_Y(y) > 0 \\ 0 & \text{otherwise} \end{cases},$$

$$h(y) = \begin{cases} \mathrm{Var}[X \mid Y=y] & \text{if } f_Y(y) > 0 \\ 0 & \text{otherwise} \end{cases}$$

be well-defined piecewise cts functions. Then

(i) $E[g(Y)] = E[X]$

(ii) $\mathrm{Var}[X] = E[h(Y)] + \mathrm{Var}[g(Y)]$

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be well-defined piecewise cts functions. Then

(i) $E[g(Y)] = E[X]$ $\quad E[E[X|Y]] = E[X]$

(ii) $Var[X] = E[h(Y)] + Var[g(Y)]$

$$E[Var[X|Y]] + Var[E[X|Y]] = Var[X]$$

"**Proof of (1)**"

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x f_{X|Y=y}(x)\, dx \right\} f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \frac{f(x,y)}{f_Y(y)} f_Y(y)\, dy \quad dx$$

$$= \int_{-\infty}^{\infty} x \left\{ \int_{-\infty}^{\infty} f(x,y)\, dy \right\} dx = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= E[X]$$

Example.

$$X \sim \text{Uniform}(0,1)$$

$$Y \sim \text{Uniform}(0,1)$$

X and Y independent

$$U = \min\{X, Y\}, \quad V = \max\{X, Y\}$$

$$f_U(u) = \begin{cases} 2(1-u), & 0<u<1 \\ 0 & \text{otherwise} \end{cases};$$

$$f_V(v) = \begin{cases} 2v, & 0<v<1 \\ 0, & \text{otherwise} \end{cases}$$

The joint distribution was found as

$$f(u,v) = \begin{cases} 2 & \text{if } 0<u<v<1 \\ 0 & \text{otherwise} \end{cases}$$

Let us find the conditional density

$$f_{V|U=u}(v)$$

$$= \frac{f(u,v)}{f_U(u)} = \begin{cases} \frac{1}{1-u} & \text{if } u<v<1 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore V|U=u \sim \text{Uniform}(u,1)$.

Moment Generating Function

Moment Generating Function

Suppose X is a random variable. For a positive integer k,

$m_k = E[X^k]$ is known as the

k^{th} moment of X.

Theorem Let X be a random variable. Let $k \in \mathbb{N}$. If $E[X^k] < \infty$ then $E[X^j] < \infty$ for $j \in \mathbb{N}$, $j \leq k$.

Prove by considering $|x| < 1$ and $|x| > 1$ separately.

Defn

Suppose X is a random variable.

and $D = \{t \in \mathbb{R} : E[e^{tX}] \text{ exists}\}$

The function $M : D \to \mathbb{R}$ given by

$M(t) = E[e^{tX}]$ is called the

moment generating function.

For a discrete r.v. $X : \Omega \to T$

$T = \{x_i : i \in \mathbb{N}\}$, for $t \in D$

$$\boxed{M_X(t) = \sum_{i \geq 1} e^{tx_i} P(X = x_i)}$$

whereas for a cts r.v.

$$\boxed{M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx}$$

<u>Example 1</u>

$$X \sim \text{Poisson}(\lambda)$$

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk} P(X=k)$$

$$= \sum_{k=0}^{\infty} \frac{e^{tk} e^{-\lambda} \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k}{k!}$$

$$= e^{-\lambda} e^{e^t \lambda} = e^{-\lambda(1-e^t)}$$

Example 2

$$X \sim \text{Geometric}(p)$$

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{n=1}^{\infty} e^{tn} P(X=n)$$

$$= \sum_{n=1}^{\infty} e^{tn}\, p \cdot (1-p)^{n-1}$$

$$= pe^t \sum_{n=1}^{\infty} \left(e^t(1-p)\right)^{n-1}$$

$$= \frac{pe^t}{1-e^t(1-p)}$$

Example 3

$X \sim \text{Normal}(\mu, \sigma^2)$

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{\sigma^2 \cdot 2}} \, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - (2\mu x + 2\sigma^2 t x) + \mu^2)/2\sigma^2} \, dx$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

__Theorem__ Suppose for a r.v. X, $\exists\ \delta > 0$ s.t. $M_X(t)$ exists for $(-\delta, \delta)$.

(i) $$E[X^k] = M_X^{(k)}(0)$$

(ii) $$M_{aX}(t) = M_X(at)\ ,\ \text{for } a \neq 0,\ at \in (-\delta, \delta)$$

(iii) $$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$ for another independent r.v. Y whose m.g.f. also exists for $(-\delta, \delta)$.

"Proof"

$$M(t)$$

$$= E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} f(x)\,dx$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{-\infty}^{\infty} x^k f(x)\,dx = \sum_{k=0}^{\infty} \frac{t^k}{k!} E[X^k]$$

"Proof"

$$M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} f(x)\,dx$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{-\infty}^{\infty} x^k f(x)\,dx = \sum_{k=0}^{\infty} \frac{t^k}{k!} E[X^k]$$

$$\therefore \underline{M^{(k)}(0) = E[X^k]}$$

A valid proof would require advanced analysis which is not in the scope of this course.

Exercise

Use the previous theorem to find expected value and variance for Poisson, Geometric, Binomial, Normal distributions.

Suppose X and Y are random variables.

The function

$M(s,t) = E[e^{sX+tY}]$ is called

the joint m.g.f. for X and Y.

Theorem

(I) Suppose X and Y are r.v.s and $M_X(t) = M_Y(t)$ in an open interval containing zero. Then X and Y have the same distribution.

(II) Suppose (X, W) and (Y, Z) are pairs of r.v.s and suppose $M_{X,W}(s,t) = M_{Y,Z}(s,t)$ in some rectangle around the origin, then (X, W) & (Y, Z) have the same joint distrib.

Theorem Suppose that (X, Y) are a pair of continuous r.v. with joint m.g.f. $M(s,t)$.

X and Y are independent if and only if

$$M(s,t) = M_X(s)\, M_Y(t).$$

Example. let $X \sim \text{Normal}(\mu_1, \sigma_1^2)$ &

$Y \sim \text{Normal}(\mu_2, \sigma_2^2)$ be

independent.

then

$$M_X(at)M_Y(bt) = e^{a\mu_1 t + \frac{a^2\sigma_1^2 t^2}{2}} \times e^{b\mu_2 t + \frac{b^2\sigma_2^2 t^2}{2}}$$

$$= e^{(a\mu_1 + b\mu_2)t + \frac{a^2\sigma_1^2 + b^2\sigma_2^2}{2} t^2}$$

$\therefore\ aX + bY \sim \text{Normal}(a\mu_1 + b\mu_2,\ a^2\sigma_1^2 + b^2\sigma_2^2)$

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$\therefore\ aX + bY \sim \text{Normal}(a\mu_1 + b\mu_2,\ a^2\sigma_1^2 + b^2\sigma_2^2)$

One can do this for n independent copies.

Exercises.

Let $X_1, X_2, \dots, X_n$ be i.i.d. rvs

Let $Y = X_1 + X_2 + \cdots + X_n$. Prove that

$$\boxed{M_Y(t) = [M_{X_1}(t)]^n}$$

Let $Z = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$. Prove that

$$M_Z(t) = [M_{X_1}(t/n)]^n.$$

Thank you